Regional Competition For Advanced Students 2006

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by valerie

1 Let $0 < x < y$ be real numbers. Let $H = \frac{2xy}{x+y}$, $G = \sqrt{xy}$, $A = \frac{x+y}{2}$, $Q = \sqrt{\frac{x^2+y^2}{2}}$ be the harmonic, geometric, arithmetic and root mean square (quadratic mean) of x and y. As generally known $H < G < A < Q$. Arrange the intervals $[H, G]$, $[G, A]$ and $[A, Q]$ in ascending order by their length.

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2 Let $n > 1$ be a positive integer an a a real number. Determine all real solutions $(x_1, x_2, \dots, x_n)$ to following system of equations: $x_1 + ax_2 = 0$ $x_2 + a^2x_3 = 0$
$x_k + a^kx_{k+1} = 0$
$x_n + a^nx_1 = 0$

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3 In a non isosceles triangle ABC let w be the angle bisector of the exterior angle at C. Let D be the point of intersection of w with the extension of AB. Let k_A be the circumcircle of the triangle ADC and analogy k_B the circumcircle of the triangle BDC. Let t_A be the tangent line to k_A in A and t_B the tangent line to k_B in B. Let P be the point of intersection of t_A and t_B. Given are the points A and B. Determine the set of points $P = P(C)$ over all points C, so that ABC is a non isosceles, acute-angled triangle.

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4 Let $< h_n > n \in \mathbb{N}$ a harmonic sequence of positive real numbers (that means that every h_n is the harmonic mean of its two neighbours h_{n-1} and h_{n+1} : $h_n = \frac{2h_{n-1}h_{n+1}}{h_{n-1}+h_{n+1}}$)
Show that: if the sequence includes a member h_j, which is the square of a rational number, it includes infinitely many members h_k, which are squares of rational numbers.
