Regional Competition For Advanced Students 2011
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by Martin N.

1 Let $p_1, p_2, \dots, p_{42}$ be 42 pairwise distinct prime numbers. Show that the sum

$$\sum_{j=1}^{42} \frac{1}{p_j^2+1}$$

is not a unit fraction $\frac{1}{n^2}$ of some integer square number.

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2 Determine all triples (x, y, z) of real numbers such that the following system of equations holds true:

$$2^{\sqrt[3]{x^2}} \cdot 4^{\sqrt[3]{y^2}} \cdot 16^{\sqrt[3]{z^2}} = 128$$
$$\left(xy^2+z^4\right)^2 = 4 + \left(xy^2-z^4\right)^2.$$

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3 Let k be a circle centered at M and let t be a tangentline to k through some point $T \in k$. Let P be a point on t and let $g \neq t$ be a line through P intersecting k at U and V. Let S be the point on k bisecting the arc UV not containing T and let Q be the the image of P under a reflection over ST.
Prove that Q, T, U and V are vertices of a trapezoid.

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4 Define the sequence $(a_n)_{n=1}^{\infty}$ of positive integers by $a_1 = 1$ and the condition that a_{n+1} is the least integer such that

$$\mathrm{lcm}(a_1, a_2, \dots, a_{n+1}) > \mathrm{lcm}(a_1, a_2, \dots, a_n).$$

Determine the set of elements of (a_n).
