Federal Competition For Advanced Students, Part 1 2010

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by Martin N.

1 Let $f(n) = \sum_{k=0}^{2010} n^k$. Show that for any integer m satisfying $2 \leqslant m \leqslant 2010$, there exists no natural number n such that $f(n)$ is divisible by m.

(41st Austrian Mathematical Olympiad, National Competition, part 1, Problem 1)

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2 For a positive integer n, we define the function $f_n(x) = \sum_{k=1}^{n} |x-k|$ for all real numbers x. For any two-digit number n (in decimal representation), determine the set of solutions $\mathbb{L}_n$ of the inequality $f_n(x) < 41$.

(41st Austrian Mathematical Olympiad, National Competition, part 1, Problem 2)

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3 Given is the set $M_n = \{0, 1, 2, \dots, n\}$ of nonnegative integers less than or equal to n. A subset S of M_n is called *outstanding* if it is non-empty and for every natural number $k \in S$, there exists a k-element subset T_k of S.
Determine the number $a(n)$ of outstanding subsets of M_n.

(41st Austrian Mathematical Olympiad, National Competition, part 1, Problem 3)

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4 The the parallel lines through an inner point P of triangle $\triangle ABC$ split the triangle into three parallelograms and three triangles adjacent to the sides of $\triangle ABC$.
(a) Show that if P is the incenter, the perimeter of each of the three small triangles equals the length of the adjacent side.
(b) For a given triangle $\triangle ABC$, determine all inner points P such that the perimeter of each of the three small triangles equals the length of the adjacent side.
(c) For which inner point does the sum of the areas of the three small triangles attain a minimum?

(41st Austrian Mathematical Olympiad, National Competition, part 1, Problem 4)
