Federal Competition For Advanced Students, Part 1 2011

www.artofproblemsolving.com/community/c3781

by Martin N.

1 Determine all integer triplets (x, y, z) such that

$$x^4 + x^2 = 7^z y^2.$$

---

2 For a positive integer k and real numbers x and y, let

$$f_k(x, y) = (x + y) - \left(x^{2k+1} + y^{2k+1}\right).$$

If $x^2 + y^2 = 1$, then determine the maximal possible value c_k of $f_k(x, y)$.

---

3 A set of three elements is called arithmetic if one of its elements is the arithmetic mean of the other two. Likewise, a set of three elements is called harmonic if one of its elements is the harmonic mean of the other two.

How many three-element subsets of the set of integers $\{z \in \mathbb{Z} \mid -2011 < z < 2011\}$ are arithmetic and harmonic?

(Remark: The arithmetic mean $A(a, b)$ and the harmonic mean $H(a, b)$ are defined as

$$A(a, b) = \frac{a + b}{2} \quad \text{and} \quad H(a, b) = \frac{2ab}{a + b} = \frac{2}{\frac{1}{a} + \frac{1}{b}},$$

respectively, where $H(a, b)$ is not defined for some a, b.)

---

4 Inside or on the faces of a tetrahedron with five edges of length 2 and one edge of lenght 1, there is a point P having distances a, b, c, d to the four faces of the tetrahedron. Determine the locus of all points P such that $a + b + c + d$ is minimal and the locus of all points P such that $a + b + c + d$ is maximal.
