Ukraine Team Selection Test 2007

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1 $\{a,b,c\} \subset \left(\frac{1}{\sqrt{6}}, +\infty\right)$ such that $a^2+b^2+c^2=1$. Prove that $\frac{1+a^2}{\sqrt{2a^2+3ab-c^2}} + \frac{1+b^2}{\sqrt{2b^2+3bc-a^2}} + \frac{1+c^2}{\sqrt{2c^2+3ca-b^2}} \geq 2(a+b+c)$.

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2 $ABCD$ is convex $AD \parallel BC$, $AC \perp BD$. M is interior point of $ABCD$ which is not a intersection of diagonals AC and BD such that $\angle AMB = \angle CMD = \frac{\pi}{2}$ .P is intersection of angel bisectors of $\angle A$ and $\angle C$. Q is intersection of angel bisectors of $\angle B$ and $\angle D$. Prove that $\angle PMB = \angle QMC$.

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3 It is known that k and n are positive integers and

$$k+1 \leq \sqrt{\frac{n+1}{\ln(n+1)}}.$$

Prove that there exists a polynomial $P(x)$ of degree n with coefficients in the set $\{0,1,-1\}$ such that $(x-1)^k$ divides $P(x)$.

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4 Find all functions $f:\mathbb{Q}\to\mathbb{Q}$ such that $f(x^2+y+f(xy)) = 3+(x+f(y)-2)f(x)$ for all $x,y\in\mathbb{Q}$.

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5 AA_3 and BB_3 are altitudes of acute-angled $\triangle ABC$. Points A_1 and B_1 are second points of intersection lines AA_3 and BB_3 with circumcircle of $\triangle ABC$ respectively. A_2 and B_2 are points on BC and AC respectively. $A_1A_2 \parallel AC$, $B_1B_2 \parallel BC$. Point M is midpoint of A_2B_2. $\angle BCA = x$. Find $\angle A_3MB_3$.

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6 Find all primes p for that there is an integer n such that there are no integers x,y with $x^3+y^3 \equiv n \mod p$ (so not all residues are the sum of two cubes).

E.g. for $p=7$, one could set $n=\pm 3$ since $x^3,y^3 \equiv 0,\pm 1 \mod 7$, thus $x^3+y^3 \equiv 0,\pm 1,\pm 2 \mod 7$ only.

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7 There are 25 people. Every two of them are use some language to speak between. They use only one language even if they both know another one. Among every three of them there is one who speaking with two other on the same language. Prove that there exist one who speaking with 10 other on the same language.

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8 $F(x)$ is polynomial with real coefficients. $F(x) = x^4+a_1x^3+a_2x^2+a_1x^1+a_0$. M is local maximum and m is minimum. Prove that $\frac{3}{10}(\frac{a_1^2}{4}-\frac{2a_2}{3^2})^2 < M-m < 3(\frac{a_1^2}{4}-\frac{2a_2}{3^2})^2$

9 Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

10 Find all positive integers n such that acute-angled $\triangle ABC$ with $\angle BAC < \frac{\pi}{4}$ could be divided into n quadrilateral. Every quadrilateral is inscribed in circle and radiuses of circles are in geometric progression.
be carefull ! :lol:

11 We have $n \geq 2$ lamps $L_1, ..., L_n$ in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows: if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off; otherwise, L_i is switched on.
Initially all the lamps are off except the leftmost one which is on.
(a) Prove that there are infinitely many integers n for which all the lamps will eventually be off.
(b) Prove that there are infinitely many integers n for which the lamps will never be all off.

12 Prove that there are infinitely many positive integers n for which all the prime divisors of $n^2 + n + 1$ are not more then $\sqrt{n}$.

Stronger one.
Prove that there are infinitely many positive integers n for which all the prime divisors of $n^3 - 1$ are not more then $\sqrt{n}$.