AoPS Online
Art of Problem Solving

Ukraine Team Selection Test 2008

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by delegat, April, Yulia, the.sceth, orl

1 Denote by M midpoint of side BC in an isosceles triangle $\triangle ABC$ with $AC = AB$. Take a point X on a smaller arc $\overset{\frown}{MA}$ of circumcircle of triangle $\triangle ABM$. Denote by T point inside of angle BMA such that $\angle TMX = 90$ and $TX = BX$.

Prove that $\angle MTB - \angle CTM$ does not depend on choice of X.

Author: Farzan Barekat, Canada

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2 There is a row that consists of digits from 0 to 9 and Ukrainian letters (there are 33 of them) with following properties: there arent two distinct digits or letters a_i, a_j such that $a_i > a_j$ and $i < j$ (if a_i, a_j are letters $a_i > a_j$ means that a_i has greater then a_j position in alphabet) and there arent two equal consecutive symbols or two equal symbols having exactly one symbol between them. Find the greatest possible number of symbols in such row.

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3 For positive a, b, c, d prove that $(a+b)(b+c)(c+d)(d+a)(1+\sqrt[4]{abcd})^4 \geq 16abcd(1+a)(1+b)(1+c)(1+d)$

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4 Two circles ω_1 and ω_2 tangents internally in point P. On their common tangent points A, B are chosen such that P lies between A and B. Let C and D be the intersection points of tangent from A to ω_1, tangent from B to ω_2 and tangent from A to ω_2, tangent from B to ω_1, respectively. Prove that $CA + CB = DA + DB$.

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5 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $f(x + f(y)) = f(x + y) + f(y)$ for all pairs of positive reals x and y. Here, $\mathbb{R}^+$ denotes the set of all positive reals.

Proposed by Paisan Nakmahachalasint, Thailand

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6 Prove that there exist infinitely many pairs (a, b) of natural numbers not equal to 1 such that $b^b + a$ is divisible by $a^a + b$.

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7 There is graph G_0 on vertices $A_1, A_2, \dots, A_n$. Graph G_{n+1} on vertices $A_1, A_2, \dots, A_n$ is constructed by the rule: A_i and A_j are joined only if in graph G_n there is a vertices $A_k \neq A_i, A_j$ such that A_k is joined with both A_i and A_j. Prove that the sequence $\{G_n\}_{n\in\mathbb{N}}$ is periodic after some term with period $T \leq 2^n$.

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8 Consider those functions $f : \mathbb{N} \mapsto \mathbb{N}$ which satisfy the condition

$$f(m+n) \geq f(m) + f(f(n)) - 1$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

Author: Nikolai Nikolov, Bulgaria

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9 Given $\triangle ABC$ with point D inside. Let $A_0 = AD \cap BC$, $B_0 = BD \cap AC$, $C_0 = CD \cap AB$ and A_1, B_1, C_1, A_2, B_2, C_2 are midpoints of BC, AC, AB, AD, BD, CD respectively. Two lines parallel to A_1A_2 and C_1C_2 and passes through point B_0 intersects B_1B_2 in points A_3 and C_3respectively. Prove that $\frac{A_3B_1}{A_3B_2} = \frac{C_3B_1}{C_3B_2}$.

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10 Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

Author: Dan Brown, Canada

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11 Let $ABCDE$ be convex pentagon such that $S(ABC) = S(BCD) = S(CDE) = S(DEA) = S(EAB)$. Prove that there is a point M inside pentagon such that $S(MAB) = S(MBC) = S(MCD) = S(MDE) = S(MEA)$.

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12 Prove that for all natural m, n polynomial $\sum_{i=0}^{m} \binom{n+i}{n} \cdot x^i$ has at most one real root.
