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Day 1 March 19

1 Find all integers $n \geq 3$ such that the following property holds: if we list the divisors of $n!$ in increasing order as $1 = d_1 < d_2 < \cdots < d_k = n!$, then we have

$$d_2 - d_1 \leq d_3 - d_2 \leq \cdots \leq d_k - d_{k-1}.$$

Proposed by Luke Robitaille.

2 Let $S_1, S_2, \ldots, S_{100}$ be finite sets of integers whose intersection is not empty. For each non-empty $T \subseteq \{S_1, S_2, \ldots, S_{100}\}$, the size of the intersection of the sets in T is a multiple of the number of sets in T. What is the least possible number of elements that are in at least 50 sets?

Proposed by Rishabh Das

3 Let m be a positive integer. A triangulation of a polygon is [i]m-balanced[/i] if its triangles can be colored with m colors in such a way that the sum of the areas of all triangles of the same color is the same for each of the m colors. Find all positive integers n for which there exists an m-balanced triangulation of a regular n-gon.

Note: A triangulation of a convex polygon $\mathcal{P}$ with $n \geq 3$ sides is any partitioning of $\mathcal{P}$ into $n-2$ triangles by $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the polygon's interior.

Proposed by Krit Boonsiriseth

Day 2 March 20

4 Let m and n be positive integers. A circular necklace contains mn beads, each either red or blue. It turned out that no matter how the necklace was cut into m blocks of n consecutive beads, each block had a distinct number of red beads. Determine, with proof, all possible values of the ordered pair (m, n).

Proposed by Rishabh Das

5 Point D is selected inside acute $\triangle ABC$ so that $\angle DAC = \angle ACB$ and $\angle BDC = 90^\circ + \angle BAC$. Point E is chosen on ray BD so that $AE = EC$. Let M be the midpoint of BC.

Show that line AB is tangent to the circumcircle of triangle BEM.

Proposed by Anton Trygub

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6 Let $n > 2$ be an integer and let $\ell \in \{1, 2, \dots, n\}$. A collection $A_1, \dots, A_k$ of (not necessarily distinct) subsets of $\{1, 2, \dots, n\}$ is called ℓ-large if $|A_i| \geq \ell$ for all $1 \leq i \leq k$. Find, in terms of n and ℓ, the largest real number c such that the inequality

$$\sum_{i=1}^{k} \sum_{j=1}^{k} x_i x_j \frac{|A_i \cap A_j|^2}{|A_i| \cdot |A_j|} \geq c \left(\sum_{i=1}^{k} x_i \right)^2$$

holds for all positive integer k, all nonnegative real numbers $x_1, x_2, \dots, x_k$, and all ℓ-large collections $A_1, A_2, \dots, A_k$ of subsets of $\{1, 2, \dots, n\}$.

Proposed by Titu Andreescu and Gabriel Dospinescu

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