IMO 1974

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Day 1

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1 Three players A, B and C play a game with three cards and on each of these 3 cards it is written a positive integer, all 3 numbers are different. A game consists of shuffling the cards, giving each player a card and each player is attributed a number of points equal to the number written on the card and then they give the cards back. After a number (≥ 2) of games we find out that A has 20 points, B has 10 points and C has 9 points. We also know that in the last game B had the card with the biggest number. Who had in the first game the card with the second value (this means the middle card concerning its value).

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2 Let ABC be a triangle. Prove that there exists a point D on the side AB of the triangle ABC, such that CD is the geometric mean of AD and DB, iff the triangle ABC satisfies the inequality $\sin A \sin B \leq \sin^2 \frac{C}{2}$.

Alternative formulation, from IMO ShortList 1974, Finland 2: We consider a triangle ABC. Prove that: $\sin(A)\sin(B) \leq \sin^2\left(\frac{C}{2}\right)$ is a necessary and sufficient condition for the existence of a point D on the segment AB so that CD is the geometrical mean of AD and BD.

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3 Prove that for any n natural, the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$$

cannot be divided by 5.

Day 2

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4 Consider decompositions of an 8×8 chessboard into p non-overlapping rectangles subject to the following conditions:
(i) Each rectangle has as many white squares as black squares.
(ii) If a_i is the number of white squares in the i-th rectangle, then $a_1 < a_2 < \ldots < a_p$.
Find the maximum value of p for which such a decomposition is possible. For this value of p, determine all possible sequences $a_1, a_2, \ldots, a_p$.

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5 The variables a, b, c, d, traverse, independently from each other, the set of positive real values. What are the values which the expression

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

takes?

6 Let $P(x)$ be a polynomial with integer coefficients. We denote $\deg(P)$ its degree which is ≥ 1. Let $n(P)$ be the number of all the integers k for which we have $(P(k))^2 = 1$. Prove that $n(P) - \deg(P) \leq 2$.