IMO 1976

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Day 1

1 In a convex quadrilateral (in the plane) with the area of 32 cm^2 the sum of two opposite sides and a diagonal is 16 cm. Determine all the possible values that the other diagonal can have.

2 Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for j$= 2, \dots$ Prove that for any positive integer n the roots of the equation $P_n(x) = x$ are all real and distinct.

3 A box whose shape is a parallelepiped can be completely filled with cubes of side 1. If we put in it the maximum possible number of cubes, each of volume 2, with the sides parallel to those of the box, then exactly 40 percent of the volume of the box is occupied. Determine the possible dimensions of the box.

Day 2

1 Determine the greatest number, who is the product of some positive integers, and the sum of these numbers is 1976.

2 We consider the following system
with $q = 2p$:

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1q}x_q = 0, \\ a_{21}x_1 + \ldots + a_{2q}x_q = 0, \\ \ldots, \\ a_{p1}x_1 + \ldots + a_{pq}x_q = 0, \end{aligned}$$

in which every coefficient is an element from the set $\{-1, 0, 1\}$. Prove that there exists a solution $x_1, \ldots, x_q$ for the system with the properties:

a.) all $x_j, j = 1, \ldots, q$ are integers;

b.) there exists at least one j for which $x_j \neq 0$;

c.) $|x_j| \leq q$ for any $j = 1, \ldots, q$.

3 A sequence (u_n) is defined by

$$u_0 = 2 \quad u_1 = \frac{5}{2}, u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1 \quad \text{for } n = 1, \ldots$$

Prove that for any positive integer n we have

$$[u_n] = 2^{\frac{(2^n-(-1)^n)}{3}}$$

(where $[x]$ denotes the smallest integer $\leq x$)