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by orl, enescu

Day 1

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1 In the interior of a square $ABCD$ we construct the equilateral triangles $ABK, BCL, CDM, DAN.$ Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments $AK, BK, BL, CL, CM, DM, DN, AN$ are the 12 vertices of a regular dodecagon.

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2 In a finite sequence of real numbers the sum of any seven successive terms is negative and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

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3 Let n be a given number greater than 2. We consider the set V_n of all the integers of the form $1 + kn$ with $k = 1, 2, \ldots$ A number m from V_n is called indecomposable in V_n if there are not two numbers p and q from V_n so that $m = pq$. Prove that there exist a number $r \in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Expressions which differ only in order of the elements of V_n will be considered the same.)

Day 2

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1 Let a, b, A, B be given reals. We consider the function defined by

$$f(x) = 1 - a \cdot \cos(x) - b \cdot \sin(x) - A \cdot \cos(2x) - B \cdot \sin(2x).$$

Prove that if for any real number x we have $f(x) \geq 0$ then $a^2 + b^2 \leq 2$ and $A^2 + B^2 \leq 1.$

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2 Let a, b be two natural numbers. When we divide $a^2 + b^2$ by $a + b$, we the the remainder r and the quotient q. Determine all pairs (a, b) for which $q^2 + r = 1977.$

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3 Let $\mathbb{N}$ be the set of positive integers. Let f be a function defined on $\mathbb{N}$, which satisfies the inequality $f(n + 1) > f(f(n))$ for all $n \in \mathbb{N}$. Prove that for any n we have $f(n) = n.$
