IMO 1978

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Day 1

1 Let m and n be positive integers such that $1 \leq m < n$. In their decimal representations, the last three digits of 1978^m are equal, respectively, to the last three digits of 1978^n. Find m and n such that $m + n$ has its least value.

2 We consider a fixed point P in the interior of a fixed sphere. We construct three segments PA, PB, PC, perpendicular two by two, with the vertexes A, B, C on the sphere. We consider the vertex Q which is opposite to P in the parallelepiped (with right angles) with PA, PB, PC as edges. Find the locus of the point Q when A, B, C take all the positions compatible with our problem.

3 Let $0 < f(1) < f(2) < f(3) < \ldots$ a sequence with all its terms positive. The $n-th$ positive integer which doesn't belong to the sequence is $f(f(n)) + 1$. Find $f(240)$.

Day 2

1 In a triangle ABC we have $AB = AC$. A circle which is internally tangent with the circumscribed circle of the triangle is also tangent to the sides AB, AC in the points P, respectively Q. Prove that the midpoint of PQ is the center of the inscribed circle of the triangle ABC.

2 Let f be an injective function from $1, 2, 3, \ldots$ in itself. Prove that for any n we have: $\sum_{k=1}^n f(k)k^{-2} \geq \sum_{k=1}^n k^{-1}$.

3 An international society has its members from six different countries. The list of members contain 1978 names, numbered $1, 2, \ldots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.