IMO 1981

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Day 1

1 Consider a variable point P inside a given triangle ABC. Let D, E, F be the feet of the perpendiculars from the point P to the lines BC, CA, AB, respectively. Find all points P which minimize the sum

$$\frac{BC}{PD}+\frac{CA}{PE}+\frac{AB}{PF}.$$

2 Take r such that $1 \le r \le n$, and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each subset has a smallest element. Let $F(n, r)$ be the arithmetic mean of these smallest elements. Prove that:

$$F(n,r)=\frac{n+1}{r+1}.$$

3 Determine the maximum value of m^2+n^2, where m and n are integers in the range $1, 2, \ldots, 1981$ satisfying $(n^2-mn-m^2)^2=1$.

Day 2

1 **a.)** For which $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n-1$ numbers?

b.) For which $n > 2$ is there exactly one set having this property?

2 Three circles of equal radius have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point O.

3 The function $f(x, y)$ satisfies: $f(0, y)=y+1$, $f(x+1, 0)=f(x, 1)$, $f(x+1, y+1)=f(x, f(x+1, y))$ for all non-negative integers x, y. Find $f(4, 1981)$.