IMO 1982

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Day 1

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1 The function $f(n)$ is defined on the positive integers and takes non-negative integer values. $f(2) = 0, f(3) > 0, f(9999) = 3333$ and for all m, n :

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1.$$

Determine $f(1982)$.

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2 A non-isosceles triangle $A_1A_2A_3$ has sides a_1, a_2, a_3 with the side a_i lying opposite to the vertex A_i. Let M_i be the midpoint of the side a_i, and let T_i be the point where the inscribed circle of triangle $A_1A_2A_3$ touches the side a_i. Denote by S_i the reflection of the point T_i in the interior angle bisector of the angle A_i. Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.

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3 Consider infinite sequences $\{x_n\}$ of positive reals such that $x_0 = 1$ and $x_0 \geq x_1 \geq x_2 \geq \ldots$.

a) Prove that for every such sequence there is an $n \geq 1$ such that:

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

b) Find such a sequence such that for all n:

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} < 4.$$

Day 2

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1 Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers x, y, then it has at least three such solutions. Show that the equation has no solutions in integers for $n = 2891$.

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2 The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by inner points M and N respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M and N are collinear.

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3 Let S be a square with sides length 100. Let L be a path within S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ with $A_0 = A_n$. Suppose that for every point P on the boundary of S there is a point of L at a distance from P no greater than $\frac{1}{2}$. Prove that there are two points X and Y of L such that the distance between X and Y is not greater than 1 and the length of the part of L which lies between X and Y is not smaller than 198.
