IMO 1983

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Day 1

1 Find all functions f defined on the set of positive reals which take positive real values and satisfy: $f(xf(y)) = yf(x)$ for all x, y; and $f(x) \to 0$ as $x \to \infty$.

2 Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2 respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

3 Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$, where x, y, z are non-negative integers.

Day 2

1 Let ABC be an equilateral triangle and $\mathcal{E}$ the set of all points contained in the three segments AB, BC, and CA (including A, B, and C). Determine whether, for every partition of $\mathcal{E}$ into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle.

2 Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression?

3 Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Determine when equality occurs.