IMO 1984

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Day 1

1 Prove that $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$, where x, y and z are non-negative real numbers satisfying $x + y + z = 1$.

2 Find one pair of positive integers a, b such that $ab(a+b)$ is not divisible by 7, but $(a+b)^7 - a^7 - b^7$ is divisible by 7^7.

3 Given points O and A in the plane. Every point in the plane is colored with one of a finite number of colors. Given a point X in the plane, the circle $C(X)$ has center O and radius $OX + \frac{\angle AOX}{OX}$, where $\angle AOX$ is measured in radians in the range $[0, 2\pi)$. Prove that we can find a point X, not on OA, such that its color appears on the circumference of the circle $C(X)$.

Day 2

1 Let $ABCD$ be a convex quadrilateral with the line CD being tangent to the circle on diameter AB. Prove that the line AB is tangent to the circle on diameter CD if and only if the lines BC and AD are parallel.

2 Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices (where $n > 3$). Let p be its perimeter. Prove that:

$$n - 3 < \frac{2d}{p} < \left[\frac{n}{2}\right] \cdot \left[\frac{n+1}{2}\right] - 2,$$

where $[x]$ denotes the greatest integer not exceeding x.

3 Let a, b, c, d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m, then $a = 1$.