IMO 1985

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Day 1

1 A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

2 Let n and k be relatively prime positive integers with $k < n$. Each number in the set $M = \{1, 2, 3, \ldots, n-1\}$ is colored either blue or white. For each i in M, both i and $n-i$ have the same color. For each $i \neq k$ in M both i and $|i-k|$ have the same color. Prove that all numbers in M must have the same color.

3 For any polynomial $P(x) = a_0 + a_1x + \ldots + a_kx^k$ with integer coefficients, the number of odd coefficients is denoted by $o(P)$. For $i - 0, 1, 2, \ldots$ let $Q_i(x) = (1+x)^i$. Prove that if $i_1, i_2, \ldots, i_n$ are integers satisfying $0 \leq i_1 < i_2 < \ldots < i_n$, then:

$$o(Q_{i_1} + Q_{i_2} + \ldots + Q_{i_n}) \geq o(Q_{i_1}).$$

Day 2

4 Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 23, prove that M contains a subset of 4 elements whose product is the 4th power of an integer.

5 A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90^\circ$.

6 For every real number x_1, construct the sequence $x_1, x_2, \ldots$ by setting:

$$x_{n+1} = x_n(x_n + \frac{1}{n}).$$

Prove that there exists exactly one value of x_1 which gives $0 < x_n < x_{n+1} < 1$ for all n.