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Day 1

1	Let d be any positive integer not equal to $2, 5$ or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.
2	Given a point P_0 in the plane of the triangle $A_1A_2A_3$. Define $A_s = A_{s-3}$ for all $s \geq 4$. Construct a set of points $P_1, P_2, P_3, \ldots$ such that P_{k+1} is the image of P_k under a rotation center A_{k+1} through an angle 120^o clockwise for $k = 0, 1, 2, \ldots$. Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.
3	To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Day 2

1	Let A, B be adjacent vertices of a regular n-gon ($n \geq 5$) with center O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, with X remaining inside the polygon. Find the locus of X.
2	Find all functions f defined on the non-negative reals and taking non-negative real values such that: $f(2) = 0, f(x) \neq 0$ for $0 \leq x < 2$, and $f(xf(y))f(y) = f(x + y)$ for all x, y.
3	Given a finite set of points in the plane, each with integer coordinates, is it always possible to color the points red or white so that for any straight line L parallel to one of the coordinate axes the difference (in absolute value) between the numbers of white and red points on L is not greater than 1?