IMO 1990
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Day 1

1 Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E, and M intersects the lines BC and AC at F and G, respectively. If

$$\frac{AM}{AB}=t,$$

find $\frac{EG}{EF}$ in terms of t.

2 Let $n\geq 3$ and consider a set E of $2n-1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is **good** if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

3 Determine all integers $n>1$ such that

$$\frac{2^n+1}{n^2}$$

is an integer.

Day 2

1 Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f:\mathbb{Q}^+\rightarrow\mathbb{Q}^+$ such that

$$f(xf(y))=\frac{f(x)}{y}$$

for all x, y in $\mathbb{Q}^+$.

2 Given an initial integer $n_0>1$, two players, $\mathcal{A}$ and $\mathcal{B}$, choose integers n_1, n_2, n_3, $\ldots$ alternately according to the following rules :

I.) Knowing n_{2k}, $\mathcal{A}$ chooses any integer n_{2k+1} such that

$$n_{2k}\leq n_{2k+1}\leq n_{2k}^2.$$

II.) Knowing n_{2k+1}, $\mathcal{B}$ chooses any integer n_{2k+2} such that

$$\frac{n_{2k+1}}{n_{2k+2}}$$

is a prime raised to a positive integer power.

Player $\mathcal{A}$ wins the game by choosing the number 1990; player $\mathcal{B}$ wins by choosing the number 1. For which n_0 does :

a.) $\mathcal{A}$ have a winning strategy?
b.) $\mathcal{B}$ have a winning strategy?
c.) Neither player have a winning strategy?

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3 Prove that there exists a convex 1990-gon with the following two properties :

a.) All angles are equal.
b.) The lengths of the 1990 sides are the numbers 1^2, 2^2, 3^2, $\cdots$, 1990^2 in some order.
