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**Day 1**

**1**  Given a triangle $ABC$, let $I$ be the center of its inscribed circle. The internal bisectors of the angles $A, B, C$ meet the opposite sides in $A', B', C'$ respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

**2**  Let $n > 6$ be an integer and $a_1, a_2, \cdots, a_k$ be all the natural numbers less than $n$ and relatively prime to $n$. If

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that $n$ must be either a prime number or a power of $2$.

**3**  Let $S = \{1, 2, 3, \cdots, 280\}$. Find the smallest integer $n$ such that each $n$-element subset of $S$ contains five numbers which are pairwise relatively prime.

**Day 2**

**1**  Suppose $G$ is a connected graph with $k$ edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

**Note: Graph-Definition**. A **graph** consists of a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of vertices $u, v$ belongs to at most one edge. The graph $G$ is connected if for each pair of distinct vertices $x, y$ there is some sequence of vertices $x = v_0, v_1, v_2, \cdots, v_m = y$ such that each pair $v_i, v_{i+1}$ $(0 \leq i < m)$ is joined by an edge of $G$.

**2**  Let $ABC$ be a triangle and $P$ an interior point of $ABC$. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.

**3**  An infinite sequence $x_0, x_1, x_2, \ldots$ of real numbers is said to be **bounded** if there is a constant $C$ such that $|x_i| \leq C$ for every $i \geq 0$. Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that

$$|x_i - x_j||i - j|^a \geq 1$$

for every pair of distinct nonnegative integers $i, j$.