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Day 1

1 Let $n > 1$ be an integer and let $f(x) = x^n + 5 \cdot x^{n-1} + 3$. Prove that there do not exist polynomials $g(x), h(x)$, each having integer coefficients and degree at least one, such that $f(x) = g(x) \cdot h(x)$.

2 Let A, B, C, D be four points in the plane, with C and D on the same side of the line AB, such that $AC \cdot BD = AD \cdot BC$ and $\angle ADB = 90^\circ + \angle ACB$. Find the ratio

$$\frac{AB \cdot CD}{AC \cdot BD},$$

and prove that the circumcircles of the triangles ACD and BCD are orthogonal. (Intersecting circles are said to be orthogonal if at either common point their tangents are perpendicuar. Thus, proving that the circumcircles of the triangles ACD and BCD are orthogonal is equivalent to proving that the tangents to the circumcircles of the triangles ACD and BCD at the point C are perpendicular.)

3 On an infinite chessboard, a solitaire game is played as follows: at the start, we have n^2 pieces occupying a square of side $n.$ The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which n can the game end with only one piece remaining on the board?

Day 2

4 For three points A, B, C in the plane, we define $m(ABC)$ to be the smallest length of the three heights of the triangle ABC, where in the case A, B, C are collinear, we set $m(ABC) = 0$. Let A, B, C be given points in the plane. Prove that for any point X in the plane,

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

5 Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Determine if there exists a strictly increasing function $f: \mathbb{N} \mapsto \mathbb{N}$ with the following properties:

(i) $f(1) = 2$;

(ii) $f(f(n)) = f(n) + n, (n \in \mathbb{N})$.

6 Let $n > 1$ be an integer. In a circular arrangement of n lamps $L_0, \ldots, L_{n-1},$ each of of which can either ON or OFF, we start with the situation where all lamps are ON, and then carry out a sequence of steps, $Step_0, Step_1, \ldots$. If L_{j-1} (j is taken mod n) is ON then $Step_j$ changes the state of L_j (it goes from ON to OFF or from OFF to ON) but does not change the state of any of the other lamps. If L_{j-1} is OFF then $Step_j$ does not change anything at all. Show that:

(i) There is a positive integer $M(n)$ such that after $M(n)$ steps all lamps are ON again,

(ii) If n has the form 2^k then all the lamps are ON after $n^2 - 1$ steps,

(iii) If n has the form $2^k + 1$ then all lamps are ON after $n^2 - n + 1$ steps.