IMO 1994

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Day 1 July 13th

1 Let m and n be two positive integers. Let a_1, a_2, $\ldots$, a_m be m different numbers from the set $\{1, 2, \ldots, n\}$ such that for any two indices i and j with $1 \leq i \leq j \leq m$ and $a_i + a_j \leq n$, there exists an index k such that $a_i + a_j = a_k$. Show that

$$\frac{a_1 + a_2 + ... + a_m}{m} \geq \frac{n+1}{2}.$$

2 Let ABC be an isosceles triangle with $AB = AC$. M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB. Q is an arbitrary point on BC different from B and C. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear. Prove that OQ is perpendicular to EF if and only if $QE = QF$.

3 For any positive integer k, let f_k be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ whose base 2 representation contains exactly three 1s.

(a) Prove that for any positive integer m, there exists at least one positive integer k such that $f(k) = m$.

(b) Determine all positive integers m for which there exists *exactly one* k with $f(k) = m$.

Day 2 July 14th

4 Find all ordered pairs (m, n) where m and n are positive integers such that $\frac{n^3+1}{mn-1}$ is an integer.

5 Let S be the set of all real numbers strictly greater than 1. Find all functions $f : S \to S$ satisfying the two conditions:

(a) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x, y in S;

(b) $\frac{f(x)}{x}$ is strictly increasing on each of the two intervals $-1 < x < 0$ and $0 < x$.

6 Show that there exists a set A of positive integers with the following property: for any infinite set S of primes, there exist *two* positive integers m in A and n not in A, each of which is a product of k distinct elements of S for some $k \geq 2$.