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Day 1 July 24th

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1 In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard). For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares. Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let $f(m,n) = |S_1 - S_2|$.

a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

b) Prove that $f(m,n) \leq \frac{1}{2}\max\{m,n\}$ for all m and n.

c) Show that there is no constant $C \in \mathbb{R}$ such that $f(m,n) < C$ for all m and n.

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2 It is known that $\angle BAC$ is the smallest angle in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T.

Show that $AU = TB + TC$.

Alternative formulation:

Four different points A, B, C, D are chosen on a circle Γ such that the triangle BCD is not right-angled. Prove that:

(a) The perpendicular bisectors of AB and AC meet the line AD at certain points W and V, respectively, and that the lines CV and BW meet at a certain point T.

(b) The length of one of the line segments $AD, BT,$ and CT is the sum of the lengths of the other two.

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3 Let x_1, x_2, ..., x_n be real numbers satisfying the conditions:

$$\begin{cases} |x_1 + x_2 + \cdots + x_n| & = & 1 \\ |x_i| & \leq & \dfrac{n+1}{2} \quad \text{for } i = 1, 2, \ldots, n. \end{cases}$$

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}.$$

Day 2 July 25th

4 An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver matrix* if, for each $i = 1, 2, \ldots, n$, the i-th row and the i-th column together contain all elements of S. Show that:

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

5 Find all pairs (a, b) of positive integers that satisfy the equation: $a^{b^2} = b^a$.

6 For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways: 4; 2+2; 2+1+1; 1+1+1+1.

Prove that, for any integer $n \ge 3$ we have $2^{\frac{n^2}{4}} < f(2^n) < 2^{\frac{n^2}{2}}$.