IMO 1998

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Day 1 June 15th

1 A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides AB and CD meet at a unique point P inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

2 In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

3 For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that $\frac{\tau(n^2)}{\tau(n)} = m$.

Day 2 June 16th

4 Determine all pairs (x, y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

5 Let I be the incenter of triangle ABC. Let K, L and M be the points of tangency of the incircle of ABC with AB, BC and CA, respectively. The line t passes through B and is parallel to KL. The lines MK and ML intersect t at the points R and S. Prove that $\angle RIS$ is acute.

6 Determine the least possible value of $f(1998)$, where $f : \mathbb{N} \to \mathbb{N}$ is a function such that for all $m, n \in \mathbb{N}$,

$$f\left(n^2 f(m)\right) = m \left(f(n)\right)^2.$$