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Day 1

1 A set S of points from the space will be called **completely symmetric** if it has at least three elements and fulfills the condition that for every two distinct points A and B from S, the perpendicular bisector plane of the segment AB is a plane of symmetry for S. Prove that if a completely symmetric set is finite, then it consists of the vertices of either a regular polygon, or a regular tetrahedron or a regular octahedron.

2 Let $n \geq 2$ be a fixed integer. Find the least constant C such the inequality

$$\sum_{i<j} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_i x_i\right)^4$$

holds for any $x_1, \dots, x_n \geq 0$ (the sum on the left consists of $\binom{n}{2}$ summands). For this constant C, characterize the instances of equality.

3 Let n be an even positive integer. We say that two different cells of a $n \times n$ board are **neighboring** if they have a common side. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or not marked) has a marked neighboring cell.

Day 2

4 Find all the pairs of positive integers (x,p) such that p is a prime, $x \leq 2p$ and x^{p-1} is a divisor of $(p-1)^x + 1$.

5 Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B. MA and MB intersects Ω_1 in C and D. Prove that Ω_2 is tangent to CD.

6 Find all the functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.