IMO 2007
www.artofproblemsolving.com/community/c3834
by Valentin Vornicu

Day 1 July 25th

1 Real numbers a_1, a_2, ..., a_n are given. For each i, $(1 \le i \le n)$, define

$$d_i = \max\{a_j \mid 1 \le j \le i\} - \min\{a_j \mid i \le j \le n\}$$

and let $d = \max\{d_i \mid 1 \le i \le n\}$.

(a) Prove that, for any real numbers $x_1 \le x_2 \le \cdots \le x_n$,

$$\max\{|x_i - a_i| \mid 1 \le i \le n\} \ge \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \le x_2 \le \cdots \le x_n$ such that the equality holds in (*).

Author: Michael Albert, New Zealand

2 Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

Author: Charles Leytem, Luxembourg

3 In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitiors is a clique.) The number of members of a clique is called its *size*.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Author: Vasily Astakhov, Russia

Day 2 July 26th

4 In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Author: Marek Pechal, Czech Republic

---

5 Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

Author: Kevin Buzzard and Edward Crane, United Kingdom

---

6 Let n be a positive integer. Consider

$$S=\{(x,y,z)\mid x,y,z\in\{0,1,\ldots,n\},x+y+z>0\}$$

as a set of $(n+1)^3-1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0,0,0)$.

Author: Gerhard Wginger, Netherlands
