IMO 2008

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by orl, delegat

Day 1 July 16th

1 Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.

Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

Author: Andrey Gavrilyuk, Russia

2 **(a)** Prove that
$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\geq 1$$
for all real numbers x, y, z, each different from 1, and satisfying $xyz=1$.

(b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, and satisfying $xyz=1$.

Author: Walther Janous, Austria

3 Prove that there are infinitely many positive integers n such that n^2+1 has a prime divisor greater than $2n+\sqrt{2n}$.

Author: Kestutis Cesnavicius, Lithuania

Day 2 July 17th

4 Find all functions $f:(0,\infty)\mapsto(0,\infty)$ (so f is a function from the positive real numbers) such that
$$\frac{\left(f(w)\right)^2+\left(f(x)\right)^2}{f(y^2)+f(z^2)}=\frac{w^2+x^2}{y^2+z^2}$$
for all positive real numbers $w,x,y,z,$ satisfying $wx=yz.$

Author: Hojoo Lee, South Korea

5 Let n and k be positive integers with $k\geq n$ and $k-n$ an even number. Let $2n$ lamps labelled $1,2,$ ..., $2n$ be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off.

Let M be number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.

Determine $\frac{N}{M}$.

Author: Bruno Le Floch and Ilia Smilga, France

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6 Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

Author: Vladimir Shmarov, Russia
