IMO 2009

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by orl, ZetaX

Day 1 July 15th

1 Let n be a positive integer and let $a_1, a_2, a_3, \ldots, a_k$ $(k \geq 2)$ be distinct integers in the set $1, 2, \ldots, n$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

Proposed by Ross Atkins, Australia

2 Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

Proposed by Sergei Berlov, Russia

3 Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the sub-sequences

$$s_{s_1},\, s_{s_2},\, s_{s_3},\, \ldots \qquad \text{and} \qquad s_{s_1+1},\, s_{s_2+1},\, s_{s_3+1},\, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

Proposed by Gabriel Carroll, USA

Day 2 July 16th

4 Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E , respectively. Let K be the incentre of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

Jan Vonk, Belgium, Peter Vandendriessche, Belgium and Hojoo Lee, Korea

5 Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b + f(a) - 1).$$

(A triangle is non-degenerate if its vertices are not collinear.)

Proposed by Bruno Le Floch, France

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6 Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \ldots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

Proposed by Dmitry Khramtsov, Russia
