IMO 2010

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by canada, orl, mavropnevma

Day 1

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1 Find all function $f:\mathbb{R}\to\mathbb{R}$ such that for all $x,y\in\mathbb{R}$ the following equality holds

$$f(\lfloor x\rfloor y)=f(x)\lfloor f(y)\rfloor$$

where $\lfloor a\rfloor$ is greatest integer not greater than $a.$

Proposed by Pierre Bornsztein, France

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2 Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF=\angle CAE < \dfrac12\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

Proposed by Tai Wai Ming and Wang Chongli, Hong Kong

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3 Find all functions $g:\mathbb{N}\to\mathbb{N}$ such that

$$\left(g(m)+n\right)\left(g(n)+m\right)$$

is a perfect square for all $m,n\in\mathbb{N}.$

Proposed by Gabriel Carroll, USA

Day 2

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4 Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, respectively M. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

Proposed by Marcin E. Kuczma, Poland

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5 Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed

Type 1) Choose a non-empty box B_j, $1\leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1};

Type 2) Choose a non-empty box B_k, $1\leq k\leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

Proposed by Hans Zantema, Netherlands

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6 Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers, and s be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\} \text{ for all } n > s.$$

Prove there exist positive integers $\ell \leq s$ and N, such that

$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \geq N.$$

Proposed by Morteza Saghafiyan, Iran
