Georgia Team Selection Test 2005

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by lasha

Day 1 May 7th

1 1. The transformation $n \to 2n-1$ or $n \to 3n-1$, where n is a positive integer, is called the 'change' of n. Numbers a and b are called 'similar', if there exists such positive integer, that can be got by finite number of 'changes' from both a and b. Find all positive integers 'similar' to 2005 and less than 2005.

2 In triangle ABC we have $\angle ACB = 2\angle ABC$ and there exists the point D inside the triangle such that $AD = AC$ and $DB = DC$. Prove that $\angle BAC = 3\angle BAD$.

3 Let x, y, z be positive real numbers,satisfying equality $x^2 + y^2 + z^2 = 25$. Find the minimal possible value of the expression $\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}$.

Day 2 May 8th

4 Find all polynomials with real coefficients, for which the equality

$$P(2P(x)) = 2P(P(x)) + 2(P(x))^2$$

holds for any real number x.

5 Let $ABCD$ be a convex quadrilateral. Points P, Q and R are the feets of the perpendiculars from point D to lines BC, CA and AB, respectively. Prove that $PQ = QR$ if and only if the bisectors of the angles ABC and ADC meet on segment AC.

6 Let A be the subset of the set of positive integers, having the following 2 properties:

1) If a belong to A,than all of the divisors of a also belong to A;

2) If a and b, $1 < a < b$, belong to A, than $1 + ab$ is also in A;

Prove that if A contains at least 3 positive integers, than A contains all positive integers.

Day 3 May 14th

7 Determine all positive integers n, for which $2^{n-1}n + 1$ is a perfect square.

8 In a convex quadrilateral $ABCD$ the points P and Q are chosen on the sides BC and CD respectively so that $\angle BAP = \angle DAQ$. Prove that the line, passing through the orthocenters of triangles ABP and ADQ, is perpendicular to AC if and only if the triangles ABP and ADQ have the same areas.

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9 Let $a_0, a_1, \dots, a_n$ be integers, one of which is nonzero, and all of the numbers are not less than -1. Prove that if

$$a_0 + 2a_1 + 2^2a_2 + \cdots + 2^na_n = 0,$$

then $a_0 + a_1 + \cdots + a_n > 0$.

Day 4 May 15th

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10 Let a, b, c be positive numbers, satisfying $abc \geq 1$. Prove that

$$a^3 + b^3 + c^3 \geq ab + bc + ca.$$

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11 On the sides AB, BC, CD and DA of the rhombus $ABCD$, respectively, are chosen points E, F, G and H so, that EF and GH touch the incircle of the rhombus. Prove that the lines EH and FG are parallel.

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12 30 students participated in the mathematical Olympiad. Each of them was given 8 problems to solve. Jury estimated their work with the following rule:
1) Each problem was worth k points, if it wasn't solved by exactly k students;
2) Each student received the maximum possible points in each problem or got 0 in it;
Lasha got the least number of points. What's the maximal number of points he could have?
Remark: 1) means that if the problem was solved by exactly k students, than each of them got $30 - k$ points in it.
