Online Math Open Problems 2016

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– Spring

1 Let A_n denote the answer to the nth problem on this contest ($n = 1, \dots, 30$); in particular, the answer to this problem is A_1. Compute $2A_1(A_1 + A_2 + \cdots + A_{30})$.

Proposed by Yang Liu

2 Let x, y, and z be real numbers such that $x + y + z = 20$ and $x + 2y + 3z = 16$. What is the value of $x + 3y + 5z$?

Proposed by James Lin

3 A store offers packages of 12 pens for $\$10$ and packages of 20 pens for $\$15$. Using only these two types of packages of pens, find the greatest number of pens $\$173$ can buy at this store.

Proposed by James Lin

4 Given that x is a real number, find the minimum value of $f(x) = |x + 1| + 3|x + 3| + 6|x + 6| + 10|x + 10|$.

Proposed by Yannick Yao

5 Let ℓ be a line with negative slope passing through the point $(20, 16)$. What is the minimum possible area of a triangle that is bounded by the x-axis, y-axis, and ℓ?

Proposed by James Lin

6 In a round-robin basketball tournament, each basketball team plays every other basketball team exactly once. If there are 20 basketball teams, what is the greatest number of basketball teams that could have at least 16 wins after the tournament is completed?

Proposed by James Lin

7 Compute the number of ordered quadruples of positive integers (a, b, c, d) such that

$$a! \cdot b! \cdot c! \cdot d! = 24!.$$

Proposed by Michael Kural

8 Let $ABCDEF$ be a regular hexagon of side length 3. Let $X, Y,$ and Z be points on segments $AB, CD,$ and EF such that $AX = CY = EZ = 1$. The area of triangle XYZ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers such that b is not divisible by the square of any prime and $\gcd(a, c) = 1$. Find $100a + 10b + c$.

Proposed by James Lin

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9 Let $f(n) = 1 \times 3 \times 5 \times \cdots \times (2n-1)$. Compute the remainder when $f(1)+f(2)+f(3)+\cdots+f(2016)$ is divided by 100.

Proposed by James Lin

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10 Lazy Linus wants to minimize his amount of laundry over the course of a week (seven days), so he decides to wear only three different T-shirts and three different pairs of pants for the week. However, he doesn't want to look dirty or boring, so he decides to wear each piece of clothing for either two or three (possibly nonconsecutive) days total, and he cannot wear the same outfit (which consists of one T-shirt and one pair of pants) on two different (not necessarily consecutive) days. How many ways can he choose the outfits for these seven days?

Proposed by Yannick Yao

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11 For how many positive integers x less than 4032 is $x^2 - 20$ divisible by 16 and $x^2 - 16$ divisible by 20?

Proposed by Tristan Shin

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12 A *9-cube* is a nine-dimensional hypercube (and hence has 2^9 vertices, for example). How many five-dimensional faces does it have?

(An n dimensional hypercube is defined to have vertices at each of the points $(a_1, a_2, \cdots, a_n)$ with $a_i \in \{0, 1\}$ for $1 \le i \le n$)

Proposed by Evan Chen

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13 For a positive integer n, let $f(n)$ be the integer formed by reversing the digits of n (and removing any leading zeroes). For example $f(14172) = 27141$. Define a sequence of numbers $\{a_n\}_{n\ge0}$ by $a_0 = 1$ and for all $i \ge 0$, $a_{i+1} = 11a_i$ or $a_{i+1} = f(a_i)$. How many possible values are there for a_8?

Proposed by James Lin

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14 Let ABC be a triangle with $BC = 20$ and $CA = 16$, and let I be its incenter. If the altitude from A to BC, the perpendicular bisector of AC, and the line through I perpendicular to AB intersect at a common point, then the length AB can be written as $m+\sqrt{n}$ for positive integers m and n. What is $100m + n$?

Proposed by Tristan Shin

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15 Let a, b, c, d be four real numbers such that $a + b + c + d = 20$ and $ab + bc + cd + da = 16$. Find the maximum possible value of $abc + bcd + cda + dab$.

Proposed by Yannick Yao

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16 Jay is given a permutation $\{p_1, p_2, \ldots, p_8\}$ of $\{1, 2, \ldots, 8\}$. He may take two dividers and split the permutation into three non-empty sets, and he concatenates each set into a single integer. In other words, if Jay chooses a, b with $1 \leq a < b < 8$, he will get the three integers $\overline{p_1 p_2 \ldots p_a}$, $\overline{p_{a+1} p_{a+2} \ldots p_b}$, and $\overline{p_{b+1} p_{b+2} \ldots p_8}$. Jay then sums the three integers into a sum $N = \overline{p_1 p_2 \ldots p_a} + \overline{p_{a+1} p_{a+2} \ldots p_b} + \overline{p_{b+1} p_{b+2} \ldots p_8}$. Find the smallest positive integer M such that no matter what permutation Jay is given, he may choose two dividers such that $N \leq M$.

Proposed by James Lin

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17 A set $S \subseteq \mathbb{N}$ satisfies the following conditions:

(a) If $x, y \in S$ (not necessarily distinct), then $x + y \in S$.
(b) If x is an integer and $2x \in S$, then $x \in S$.

Find the number of pairs of integers (a, b) with $1 \leq a, b \leq 50$ such that if $a, b \in S$ then $S = \mathbb{N}$.

Proposed by Yang Liu

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18 Kevin is in kindergarten, so his teacher puts a 100×200 addition table on the board during class. The teacher first randomly generates distinct positive integers $a_1, a_2, \ldots, a_{100}$ in the range $[1, 2016]$ corresponding to the rows, and then she randomly generates distinct positive integers $b_1, b_2, \ldots, b_{200}$ in the range $[1, 2016]$ corresponding to the columns. She then fills in the addition table by writing the number $a_i + b_j$ in the square (i, j) for each $1 \leq i \leq 100$, $1 \leq j \leq 200$.

During recess, Kevin takes the addition table and draws it on the playground using chalk. Now he can play hopscotch on it! He wants to hop from $(1, 1)$ to $(100, 200)$. At each step, he can jump in one of 8 directions to a new square bordering the square he stands on a side or at a corner. Let M be the minimum possible sum of the numbers on the squares he jumps on during his path to $(100, 200)$ (including both the starting and ending squares). The expected value of M can be expressed in the form $\frac{p}{q}$ for relatively prime positive integers p, q. Find $p + q$.

Proposed by Yang Liu

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19 Let $\mathbb{Z}_{\geq 0}$ denote the set of nonnegative integers.

Define a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ with $f(0) = 1$ and

$$f(n) = 512^{\lfloor n/10 \rfloor} f(\lfloor n/10 \rfloor)$$

for all $n \geq 1$. Determine the number of nonnegative integers n such that the hexadecimal (base 16) representation of $f(n)$ contains no more than 2500 digits.

Proposed by Tristan Shin

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20 Define $A(n)$ as the average of all positive divisors of the positive integer n. Find the sum of all solutions to $A(n) = 42$.

Proposed by Yannick Yao

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21 Say a real number r is *repetitive* if there exist two distinct complex numbers z_1, z_2 with $|z_1| = |z_2| = 1$ and $\{z_1, z_2\} \neq \{-i, i\}$ such that

$$z_1(z_1^3 + z_1^2 + rz_1 + 1) = z_2(z_2^3 + z_2^2 + rz_2 + 1).$$

There exist real numbers a, b such that a real number r is *repetitive* if and only if $a < r \leq b$. If the value of $|a| + |b|$ can be expressed in the form $\frac{p}{q}$ for relatively prime positive integers p and q, find $100p + q$.

Proposed by James Lin

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22 Let ABC be a triangle with $AB = 5$, $BC = 7$, $CA = 8$, and circumcircle ω. Let P be a point inside ABC such that $PA : PB : PC = 2 : 3 : 6$. Let rays $\overrightarrow{AP}$, $\overrightarrow{BP}$, and $\overrightarrow{CP}$ intersect ω again at X, Y, and Z, respectively. The area of XYZ can be expressed in the form $\frac{p\sqrt{q}}{r}$ where p and r are relatively prime positive integers and q is a positive integer not divisible by the square of any prime. What is $p + q + r$?

Proposed by James Lin

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23 S be the set of all 2017^2 lattice points (x, y) with $x, y \in \{0\} \cup \{2^0, 2^1, \cdots, 2^{2015}\}$. A subset $X \subseteq S$ is called BQ if it has the following properties:

(a) X contains at least three points, no three of which are collinear.
(b) One of the points in X is $(0, 0)$.
(c) For any three distinct points $A, B, C \in X$, the orthocenter of $\triangle ABC$ is in X.
(d) The convex hull of X contains at least one horizontal line segment.

Determine the number of BQ subsets of S.

Proposed by Vincent Huang

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24 Bessie and her 2015 bovine buddies work at the Organic Milk Organization, for a total of 2016 workers. They have a hierarchy of bosses, where obviously no cow is its own boss. In other words, for some pairs of employees (A, B), B is the boss of A. This relationship satisfies an obvious condition: if B is the boss of A and C is the boss of B, then C is also a boss of A. Business has been slow, so Bessie hires an outside organizational company to partition the company into some number of groups. To promote growth, every group is one of two forms. Either no one in the group is the boss of another in the group, or for every pair of cows in the group, one is the boss of the other. Let G be the minimum number of groups needed in such a partition. Find the maximum value of G over all possible company structures.

Proposed by Yang Liu

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25 Given a prime p and positive integer k, an integer n with $0 \le n < p$ is called a (p,k)-Hofstadterian residue if there exists an infinite sequence of integers $n_0, n_1, n_2, \ldots$ such that $n_0 \equiv n$ and $n_{i+1}^k \equiv n_i \pmod p$ for all integers $i \ge 0$. If $f(p,k)$ is the number of (p,k)-Hofstadterian residues, then compute $\sum_{k=1}^{2016} f(2017,k)$.

Proposed by Ashwin Sah

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26 Let S be the set of all pairs (a,b) of integers satisfying $0 \le a,b \le 2014$. For any pairs $s_1 = (a_1,b_1), s_2 = (a_2,b_2) \in S$, define

$$s_1 + s_2 = ((a_1+a_2)_{2015}, (b_1+b_2)_{2015}) \text{ and } s_1 \times s_2 = ((a_1a_2 + 2b_1b_2)_{2015}, (a_1b_2 + a_2b_1)_{2015}),$$

where n_{2015} denotes the remainder when an integer n is divided by 2015.

Compute the number of functions $f : S \to S$ satisfying

$$f(s_1+s_2) = f(s_1) + f(s_2) \text{ and } f(s_1 \times s_2) = f(s_1) \times f(s_2)$$

for all $s_1, s_2 \in S$.

Proposed by Yang Liu

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27 Let ABC be a triangle with circumradius 2 and $\angle B - \angle C = 15^\circ$. Denote its circumcenter as O, orthocenter as H, and centroid as G. Let the reflection of H over O be L, and let lines AG and AL intersect the circumcircle again at X and Y, respectively. Define B_1 and C_1 as the points on the circumcircle of ABC such that $BB_1 \parallel AC$ and $CC_1 \parallel AB$, and let lines XY and B_1C_1 intersect at Z. Given that $OZ = 2\sqrt{5}$, then AZ^2 can be expressed in the form $m - \sqrt{n}$ for positive integers m and n. Find $100m + n$.

Proposed by Michael Ren

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28 Let N be the number of polynomials $P(x_1, x_2, \ldots, x_{2016})$ of degree at most 2015 with coefficients in the set $\{0,1,2\}$ such that $P(a_1, a_2, \cdots, a_{2016}) \equiv 1 \pmod 3$ for all $(a_1, a_2, \cdots, a_{2016}) \in \{0,1\}^{2016}$.

Compute the remainder when $v_3(N)$ is divided by 2011, where $v_3(N)$ denotes the largest integer k such that $3^k | N$.

Proposed by Yang Liu

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29 Yang the Spinning Square Sheep is a square in the plane such that his four legs are his four vertices. Yang can do two different types of *tricks*:

(a) Yang can choose one of his sides, then reflect himself over the side.
(b) Yang can choose one of his legs, then rotate 90° counterclockwise around the leg.

Yang notices that after 2016 tricks, each leg ends up in exactly the same place the leg started out in! Let there be N ways for Yang to perform his 2016 tricks. What is the remainder when N is divided by 100000?

Proposed by James Lin

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30 In triangle ABC, $AB = 3\sqrt{30} - \sqrt{10}$, $BC = 12$, and $CA = 3\sqrt{30} + \sqrt{10}$. Let M be the midpoint of AB and N be the midpoint of AC. Denote l as the line passing through the circumcenter O and orthocenter H of ABC, and let E and F be the feet of the perpendiculars from B and C to l, respectively. Let l' be the reflection of l in BC such that l' intersects lines AE and AF at P and Q, respectively. Let lines BP and CQ intersect at K. X, Y, and Z are the reflections of K over the perpendicular bisectors of sides BC, CA, and AB, respectively, and R and S are the midpoints of XY and XZ, respectively. If lines MR and NS intersect at T, then the length of OT can be expressed in the form $\frac{p}{q}$ for relatively prime positive integers p and q. Find $100p + q$.

Proposed by Vincent Huang and James Lin

– Fall

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1 Kevin is in first grade, so his teacher asks him to calculate $20+1\cdot 6+k$, where k is a real number revealed to Kevin. However, since Kevin is rude to his Aunt Sally, he instead calculates $(20+1)\cdot(6+k)$. Surprisingly, Kevin gets the correct answer! Assuming Kevin did his computations correctly, what was his answer?

Proposed by James Lin

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2 Yang has a standard 6-sided die, a standard 8-sided die, and a standard 10-sided die. He tosses these three dice simultaneously. The probability that the three numbers that show up form the side lengths of a right triangle can be expressed as $\frac{m}{n}$, for relatively prime positive integers m and n. Find $100m + n$.

Proposed by Yannick Yao

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3 In a rectangle $ABCD$, let M and N be the midpoints of sides BC and CD, respectively, such that AM is perpendicular to MN. Given that the length of AN is 60, the area of rectangle $ABCD$ is $m\sqrt{n}$ for positive integers m and n such that n is not divisible by the square of any prime. Compute $100m + n$.

Proposed by Yannick Yao

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4 Let $G = 10^{10^{100}}$ (a.k.a. a googolplex). Then

$$\log_{\left(\log_{\left(\log_{10} G\right)} G\right)} G$$

can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Determine the sum of the digits of $m+n$.

Proposed by Yannick Yao

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5 Jay notices that there are n primes that form an arithmetic sequence with common difference 12. What is the maximum possible value for n?

Proposed by James Lin

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6 For a positive integer n, define $n? = 1^n \cdot 2^{n-1} \cdot 3^{n-2} \cdots (n-1)^2 \cdot n^1$. Find the positive integer k for which $7?9? = 5?k?$.

Proposed by Tristan Shin

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7 The 2016 players in the Gensokyo Tennis Club are playing Up and Down the River. The players first randomly form 1008 pairs, and each pair is assigned to a tennis court (The courts are numbered from 1 to 1008). Every day, the two players on the same court play a match against each other to determine a winner and a loser. For $2 \leq i \leq 1008$, the winner on court i will move to court $i-1$ the next day (and the winner on court 1 does not move). Likewise, for $1 \leq j \leq 1007$, the loser on court j will move to court $j+1$ the next day (and the loser on court 1008 does not move). On Day 1, Reimu is playing on court 123 and Marisa is playing on court 876. Find the smallest positive integer value of n for which it is possible that Reimu and Marisa play one another on Day n.

Proposed by Yannick Yao

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8 For a positive integer n, define the nth triangular number T_n to be $\frac{n(n+1)}{2}$, and define the nth square number S_n to be n^2. Find the value of

$$\sqrt{S_{62} + T_{63}\sqrt{S_{61} + T_{62}\sqrt{\cdots\sqrt{S_2 + T_3\sqrt{S_1 + T_2}}}}}.$$

Proposed by Yannick Yao

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9 In quadrilateral $ABCD$, $AB = 7, BC = 24, CD = 15, DA = 20,$ and $AC = 25$. Let segments AC and BD intersect at E. What is the length of EC?

Proposed by James Lin

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10 Let $a_1 < a_2 < a_3 < a_4$ be positive integers such that the following conditions hold:

-$\gcd(a_i, a_j) > 1$ holds for all integers $1 \le i < j \le 4$.
-$\gcd(a_i, a_j, a_k) = 1$ holds for all integers $1 \le i < j < k \le 4$.

Find the smallest possible value of a_4.

Proposed by James Lin

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11 Let f be a random permutation on $\{1, 2, \dots, 100\}$ satisfying $f(1) > f(4)$ and $f(9) > f(16)$. The probability that $f(1) > f(16) > f(25)$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute $100m + n$.

Note: In other words, f is a function such that $\{f(1), f(2), \dots, f(100)\}$ is a permutation of $\{1, 2, \dots, 100\}$.

Proposed by Evan Chen

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12 For each positive integer $n \ge 2$, define $k(n)$ to be the largest integer m such that $(n!)^m$ divides $2016!$. What is the minimum possible value of $n + k(n)$?

Proposed by Tristan Shin

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13 Let $A_1B_1C_1$ be a triangle with $A_1B_1 = 16$, $B_1C_1 = 14$, and $C_1A_1 = 10$. Given a positive integer i and a triangle $A_iB_iC_i$ with circumcenter O_i, define triangle $A_{i+1}B_{i+1}C_{i+1}$ in the following way:

(a) A_{i+1} is on side B_iC_i such that $C_iA_{i+1} = 2B_iA_{i+1}$.
(b) $B_{i+1} \neq C_i$ is the intersection of line A_iC_i with the circumcircle of $O_iA_{i+1}C_i$.
(c) $C_{i+1} \neq B_i$ is the intersection of line A_iB_i with the circumcircle of $O_iA_{i+1}B_i$.

Find

$$\left(\sum_{i=1}^{\infty}[A_iB_iC_i]\right)^2.$$

Note: $[K]$ denotes the area of K.

Proposed by Yang Liu

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14 In Yang's number theory class, Michael K, Michael M, and Michael R take a series of tests. Afterwards, Yang makes the following observations about the test scores:

(a) Michael K had an average test score of 90, Michael M had an average test score of 91, and Michael R had an average test score of 92.

(b) Michael K took more tests than Michael M, who in turn took more tests than Michael R.

(c) Michael M got a higher total test score than Michael R, who in turn got a higher total test score than Michael K. (The total test score is the sum of the test scores over all tests)

What is the least number of tests that Michael K, Michael M, and Michael R could have taken combined?

Proposed by James Lin

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15 Two bored millionaires, Bilion and Trilion, decide to play a game. They each have a sufficient supply of $\$1, \$2, \$5$, and $\$10$ bills. Starting with Bilion, they take turns putting one of the bills they have into a pile. The game ends when the bills in the pile total exactly $\$1{,}000{,}000$, and whoever makes the last move wins the $\$1{,}000{,}000$ in the pile (if the pile is worth more than $\$1{,}000{,}000$ after a move, then the person who made the last move loses instead, and the other person wins the amount of cash in the pile). Assuming optimal play, how many dollars will the winning player gain?

Proposed by Yannick Yao

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16 For her zeroth project at Magic School, Emilia needs to grow six perfectly-shaped apple trees. First she plants six tree saplings at the end of Day 0. On each day afterwards, Emilia attempts to use her magic to turn each sapling into a perfectly-shaped apple tree, and for each sapling she succeeds in turning it into a perfectly-shaped apple tree that day with a probability of $\frac{1}{2}$. (Once a sapling is turned into a perfectly-shaped apple tree, it will stay a perfectly-shaped apple tree.) The expected number of days it will take Emilia to obtain six perfectly-shaped apple trees is $\frac{m}{n}$ for relatively prime positive integers m and n. Find $100m + n$.

Proposed by Yannick Yao

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17 Let n be a positive integer. S is a set of points such that the points in S are arranged in a regular 2016-simplex grid, with an edge of the simplex having n points in S. (For example, the 2-dimensional analog would have $\dfrac{n(n+1)}{2}$ points arranged in an equilateral triangle grid). Each point in S is labeled with a real number such that the following conditions hold:

(a) Not all the points in S are labeled with 0.

(b) If ℓ is a line that is parallel to an edge of the simplex and that passes through at least one point in S, then the labels of all the points in S that are on ℓ add to 0.

(c) The labels of the points in S are symmetric along any such line ℓ.

Find the smallest positive integer n such that this is possible.

Note: A regular 2016-simplex has 2017 vertices in 2016-dimensional space such that the distances between every pair of vertices are equal.

Proposed by James Lin

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18 Find the smallest positive integer k such that there exist positive integers $M, O > 1$ satisfying

$$(O \cdot M \cdot O)^k = (O \cdot M) \cdot \underbrace{(N \cdot O \cdot M) \cdot (N \cdot O \cdot M) \cdot \ldots \cdot (N \cdot O \cdot M)}_{2016\ (N \cdot O \cdot M)\text{s}},$$

where $N = O^M$.

Proposed by James Lin and Yannick Yao

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19 Let S be the set of all polynomials $Q(x, y, z)$ with coefficients in $\{0, 1\}$ such that there exists a homogeneous polynomial $P(x, y, z)$ of degree 2016 with integer coefficients and a polynomial $R(x, y, z)$ with integer coefficients so that

$$P(x, y, z)Q(x, y, z) = P(yz, zx, xy) + 2R(x, y, z)$$

and $P(1, 1, 1)$ is odd. Determine the size of S.

Note: A homogeneous polynomial of degree d consists solely of terms of degree d.

Proposed by Vincent Huang

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20 For a positive integer k, define the sequence $\{a_n\}_{n\geq 0}$ such that $a_0 = 1$ and for all positive integers n, a_n is the smallest positive integer greater than a_{n-1} for which $a_n \equiv ka_{n-1} \pmod{2017}$. What is the number of positive integers $1 \leq k \leq 2016$ for which $a_{2016} = 1 + \binom{2017}{2}$?

Proposed by James Lin

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21 Mark the Martian and Bark the Bartian live on planet Blok, in the year 2019. Mark and Bark decide to play a game on a 10×10 grid of cells. First, Mark randomly generates a subset S of $\{1, 2, \ldots, 2019\}$ with $|S| = 100$. Then, Bark writes each of the 100 integers in a different cell of the 10×10 grid. Afterwards, Bark constructs a solid out of this grid in the following way: for each grid cell, if the number written on it is n, then she stacks n $1 \times 1 \times 1$ blocks on top of one other in that cell. Let B be the largest possible surface area of the resulting solid, including the bottom of the solid, over all possible ways Bark could have inserted the 100 integers into the grid of cells. Find the expected value of B over all possible sets S Mark could have generated.

Proposed by Yang Liu

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22 Let ABC be a triangle with $AB = 3$ and $AC = 4$. It is given that there does not exist a point D, different from A and not lying on line BC, such that the Euler line of ABC coincides with the Euler line of DBC. The square of the product of all possible lengths of BC can be expressed in the form $m + n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. Find $100m + 10n + p$.

Note: For this problem, consider every line passing through the center of an equilateral triangle to be an Euler line of the equilateral triangle. Hence, if D is chosen such that DBC is an equilateral triangle and the Euler line of ABC passes through the center of DBC, then consider the Euler line of ABC to coincide with "the" Euler line of DBC.

Proposed by Michael Ren

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23 Let $\mathbb{N}$ denote the set of positive integers. Let $f:\mathbb{N}\to\mathbb{N}$ be a function such that the following conditions hold:

(a) For any $n\in\mathbb{N}$, we have $f(n)|n^{2016}$.
(b) For any $a,b,c\in\mathbb{N}$ satisfying $a^2+b^2=c^2$, we have $f(a)f(b)=f(c)$.

Over all possible functions f, determine the number of distinct values that can be achieved by $f(2014)+f(2)-f(2016)$.

Proposed by Vincent Huang

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24 Let $P(x,y)$ be a polynomial such that $\deg_x(P),\deg_y(P)\le 2020$ and

$$P(i,j)=\binom{i+j}{i}$$

over all 2021^2 ordered pairs (i,j) with $0\le i,j\le 2020$. Find the remainder when $P(4040,4040)$ is divided by 2017.

Note: $\deg_x(P)$ is the highest exponent of x in a nonzero term of $P(x,y)$. $\deg_y(P)$ is defined similarly.

Proposed by Michael Ren

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25 Let $X_1X_2X_3$ be a triangle with $X_1X_2=4, X_2X_3=5, X_3X_1=7,$ and centroid G. For all integers $n\ge 3$, define the set S_n to be the set of n^2 ordered pairs (i,j) such that $1\le i\le n$ and $1\le j\le n$. Then, for each integer $n\ge 3$, when given the points $X_1,X_2,\ldots,X_n$, randomly choose an element $(i,j)\in S_n$ and define X_{n+1} to be the midpoint of X_i and X_j. The value of

$$\sum_{i=0}^{\infty}\left(\mathbb{E}\left[X_{i+4}G^2\right]\left(\frac{3}{4}\right)^i\right)$$

can be expressed in the form $p+q\ln 2+r\ln 3$ for rational numbers p,q,r. Let $|p|+|q|+|r|=\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m+n$.

Note: $\mathbb{E}(x)$ denotes the expected value of x.

Proposed by Yang Liu

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26 Let ABC be a triangle with $BC=9$, $CA=8$, and $AB=10$. Let the incenter and incircle of ABC be I and γ, respectively, and let N be the midpoint of major arc BC of the cirucmcircle of ABC. Line NI meets the circumcircle of ABC a second time at P. Let the line through I perpendicular to AI meet segments AB, AC, and AP at C_1, B_1, and Q, respectively. Let B_2 lie on segment CQ such that line B_1B_2 is tangent to γ, and let C_2 lie on segment BQ such that line C_1C_2 tangent to γ. The length of B_2C_2 can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Determine $100m+n$.

Proposed by Vincent Huang

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27 Compute the number of monic polynomials $q(x)$ with integer coefficients of degree 12 such that there exists an integer polynomial $p(x)$ satisfying $q(x)p(x) = q(x^2)$.

Proposed by Yang Liu

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28 Let ABC be a triangle with $AB = 34, BC = 25$, and $CA = 39$. Let O, H, and ω be the circumcenter, orthocenter, and circumcircle of $\triangle ABC$, respectively. Let line AH meet ω a second time at A_1 and let the reflection of H over the perpendicular bisector of BC be H_1. Suppose the line through O perpendicular to A_1O meets ω at two points Q and R with Q on minor arc AC and R on minor arc AB. Denote $\mathcal{H}$ as the hyperbola passing through A, B, C, H, H_1, and suppose HO meets $\mathcal{H}$ again at P. Let X, Y be points with $XH \parallel AR \parallel YP, XP \parallel AQ \parallel YH$. Let P_1, P_2 be points on the tangent to $\mathcal{H}$ at P with $XP_1 \parallel OH \parallel YP_2$ and let P_3, P_4 be points on the tangent to $\mathcal{H}$ at H with $XP_3 \parallel OH \parallel YP_4$. If P_1P_4 and P_2P_3 meet at N, and ON may be written in the form $\frac{a}{b}$ where a, b are positive coprime integers, find $100a+b$.

Proposed by Vincent Huang

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29 Let n be a positive integer. Yang the Saltant Sanguivorous Shearling is on the side of a very steep mountain that is embedded in the coordinate plane. There is a blood river along the line $y = x$, which Yang may reach but is not permitted to go above (i.e. Yang is allowed to be located at $(2016, 2015)$ and $(2016, 2016)$, but not at $(2016, 2017)$). Yang is currently located at $(0, 0)$ and wishes to reach $(n, 0)$. Yang is permitted only to make the following moves:

(a) Yang may *spring*, which consists of going from a point (x, y) to the point $(x, y+1)$.
(b) Yang may *stroll*, which consists of going from a point (x, y) to the point $(x+1, y)$.
(c) Yang may *sink*, which consists of going from a point (x, y) to the point $(x, y-1)$.

In addition, whenever Yang does a *sink*, he breaks his tiny little legs and may no longer do a *spring* at any time afterwards. Yang also expends a lot of energy doing a *spring* and gets bloodthirsty, so he must visit the blood river at least once afterwards to quench his bloodthirst. (So Yang may still *spring* while bloodthirsty, but he may not finish his journey while bloodthirsty.) Let there be a_n different ways for which Yang can reach $(n, 0)$, given that Yang is permitted to pass by $(n, 0)$ in the middle of his journey. Find the 2016th smallest positive integer n for which $a_n \equiv 1 \pmod{5}$.

Proposed by James Lin

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30 Let $P_1(x), P_2(x), \ldots, P_n(x)$ be monic, non-constant polynomials with integer coefficients and let $Q(x)$ be a polynomial with integer coefficients such that

$$x^{2^{2016}} + x + 1 = P_1(x)P_2(x)\ldots P_n(x) + 2Q(x).$$

Suppose that the maximum possible value of $2016n$ can be written in the form $2^{b_1}+2^{b_2}+\cdots+2^{b_k}$ for nonnegative integers $b_1 < b_2 < \cdots < b_k$. Find the value of $b_1 + b_2 + \cdots + b_k$.

Proposed by Michael Ren