Argentina Team Selection Test 2005

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Day 1

1 Find all pairs of integers (m,n) such that an $m\times n$ board can be totally covered with 1×3 and 2×5 pieces.

2 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $\forall x,y\in\mathbb{R}$ we have

$$f(xf(x)+f(y))=f(x)^2+y$$

3 Given the triangle ABC we consider the points X,Y,Z such that the triangles ABZ,BCX,CAZ are equilateral, and they don't have intersection with ABC. Let B' be the midpoint of BC, N' the midpoint of CY, and M,N the midpoints of AZ,CX, respectively. Prove that $B'N'\perp MN$.

Day 2

1 We have 150 numbers $x_1,x_2,\cdots,x_{150}$ each of which is either $\sqrt{2}+1$ or $\sqrt{2}-1$

We calculate the following sum:
$S=x_1x_2+x_3x_4+x_5x_6+\cdots+x_{149}x_{150}$

Can we choose the 150 numbers such that $S=121$?
And what about $S=111$?

2 Let n,p be integers such that $n>1$ and p is a prime. If $n\mid p-1$ and $p\mid n^3-1$, show that $4p-3$ is a perfect square.

3 We say that a group of k boys is $n-acceptable$ if removing any boy from the group one can always find, in the other $k-1$ group, a group of n boys such that everyone knows each other. For each n, find the biggest k such that in any group of k boys that is $n-acceptable$ we must always have a group of $n+1$ boys such that everyone knows each other.