Argentina Team Selection Test 2007

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Day 1

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1 Let X, Y, Z be distinct positive integers having exactly two digits in such a way that:
$X = 10a + b$ $Y = 10b + c$ $Z = 10c + a$

(a, b, c are digits)

Find all posible values of $gcd(X, Y, Z)$

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2 Let $ABCD$ be a trapezium of parallel sides AD and BC and non-parallel sides AB and CD
Let I be the incenter of ABC. It is known that exists a point $Q \in AD$ with $Q \neq A$ and $Q \neq D$ such that if P is a point of the intersection of the bisectors of $\widehat{CQD}$ and $\widehat{CAD}$ then $PI \parallel AD$
Prove that $PI = BQ$

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3 A 3000×3000 square is tiled by dominoes (i. e. 1×2 rectangles) in an arbitrary way. Show that one can color the dominoes in three colors such that the number of the dominoes of each color is the same, and each dominoe d has at most two neighbours of the same color as d. (Two dominoes are said to be *neighbours* if a cell of one domino has a common edge with a cell of the other one.)

Day 2

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4 Find all real values of $x > 1$ which satisfy:
$\frac{x^2}{x-1} + \sqrt{x-1} + \frac{\sqrt{x-1}}{x^2} = \frac{x-1}{x^2} + \frac{1}{\sqrt{x-1}} + \frac{x^2}{\sqrt{x-1}}$

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5 Let $d_1, d_2 \ldots, d_r$ be the positive divisors of n $1 = d_1 < d_2 < \ldots < d_r = n$
If $(d_7)^2 + (d_{15})^2 = (d_{16})^2$ find all posible values of d_{17}

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6 For natural n we define $s(n)$ as the sum of digits of n (in base ten)
Does there exist a positive real constant c such that for all natural n we have
$\frac{s(n)}{s(n^2)} \leq c$?
