Argentina Team Selection Test 2009

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by uglysolutions

Day 1 May 14th

1 On a 50×50 board, the centers of several unit squares are colored black. Find the maximum number of centers that can be colored black in such a way that no three black points form a right-angled triangle.

2 Let $a_1, a_2, ..., a_{300}$ be nonnegative real numbers, with $\sum_{i=1}^{300} a_i = 1$.

Find the maximum possible value of $\sum_{i \neq j, i|j} a_i a_j$.

3 Let ABC be a triangle, B_1 the midpoint of side AB and C_1 the midpoint of side AC. Let P be the point of intersection ($\neq A$) of the circumcircles of triangles ABC_1 and AB_1C. Let Q be the point of intersection ($\neq A$) of the line AP and the circumcircle of triangle AB_1C_1.

Prove that $\frac{AP}{AQ} = \frac{3}{2}$.

Day 2 May 15th

4 Find all positive integers n such that $20^n - 13^n - 7^n$ is divisible by 309.

5 There are several contestants at a math olympiad. We say that two contestants A and B are *indirect friends* if there are contestants $C_1, C_2, ..., C_n$ such that A and C_1 are friends, C_1 and C_2 are friends, C_2 and C_3 are friends, ..., C_n and B are friends. In particular, if A and B are friends themselves, they are *indirect friends* as well.
Some of the contestants were friends before the olympiad. During the olympiad, some contestants make new friends, so that after the olympiad every contestant has at least one friend among the other contestants. We say that a contestant is *special* if, after the olympiad, he has exactly twice as indirect friends as he had before the olympiad.
Prove that the number of special contestants is less or equal than $\frac{2}{3}$ of the total number of contestants.

6 Let $n \geq 3$ be an odd integer. We denote by $[-n, n]$ the set of all integers greater or equal than $-n$ and less or equal than n.
Player A chooses an arbitrary positive integer k, then player B picks a subset of k (distinct) elements from $[-n, n]$. Let this subset be S.
If all numbers in $[-n, n]$ can be written as the sum of exactly n distinct elements of S, then

player A wins the game. If not, B wins.
Find the least value of k such that player A can always win the game.