Argentina Team Selection Test 2010

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by uglysolutions

Day 1 April 29th

1 In a football tournament there are 8 teams, each of which plays exacly one match against every other team. If a team A defeats team B, then A is awarded 3 points and B gets 0 points. If they end up in a tie, they receive 1 point each.
It turned out that in this tournament, whenever a match ended up in a tie, the two teams involved did not finish with the same final score. Find the maximum number of ties that could have happened in such a tournament.

2 Let ABC be a triangle with $AB = AC$. The incircle touches BC, AC and AB at D, E and F respectively. Let P be a point on the arc $\overset{\frown}{EF}$ that does not contain D. Let Q be the second point of intersection of BP and the incircle of ABC. The lines EP and EQ meet the line BC at M and N, respectively.
Prove that the four points P, F, B, M lie on a circle and $\frac{EM}{EN} = \frac{BF}{BP}$.

3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + xy + f(y)) = \left(f(x) + \frac{1}{2}\right)\left(f(y) + \frac{1}{2}\right)$$

holds for all real numbers x, y.

Day 2 April 30th

4 Two players, A and B, play a game on a board which is a rhombus of side n and angles of 60° and 120°, divided into $2n^2$ equilateral triangles, as shown in the diagram for $n = 4$. A uses a red token and B uses a blue token, which are initially placed in cells containing opposite corners of the board (the 60° ones). In turns, players move their token to a neighboring cell (sharing a side with the previous one). To win the game, a player must either place his token on the cell containing the other player's token, or get to the opposite corner to the one where he started. If A starts the game, determine which player has a winning strategy.

5 Let p and q be prime numbers. The sequence (x_n) is defined by $x_1 = 1$, $x_2 = p$ and $x_{n+1} = px_n - qx_{n-1}$ for all $n \geq 2$.
Given that there is some k such that $x_{3k} = -3$, find p and q.

6 Suppose $a_1, a_2, ..., a_r$ are integers with $a_i \geq 2$ for all i such that $a_1 + a_2 + ... + a_r = 2010$. Prove that the set $\{1, 2, 3, ..., 2010\}$ can be partitioned in r subsets $A_1, A_2, ..., A_r$ each with

$a_1, a_2, ..., a_r$ elements respectively, such that the sum of the numbers on each subset is divisible by 2011.
Decide whether this property still holds if we replace 2010 by 2011 and 2011 by 2012 (that is, if the set to be partitioned is $\{1, 2, 3, ..., 2011\}$).
