Argentina Team Selection Test 2011

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Day 1

1 Each number from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is either colored red or blue, following these rules:

a) The number 4 is colored red, and there is at least one number colored blue.
b) If two numbers x, y have different colors and $x + y \leq 8$, then the number $x + y$ is colored blue.
c) If two numbers x, y have different colors and $x \cdot y \leq 8$, then the number $x \cdot y$ is colored red.

Find all possible ways the numbers from this set can be colored.

2 A wizard kidnaps 31 members from party A, 28 members from party B, 23 members from party C, and 19 members from party D, keeping them isolated in individual rooms in his castle, where he forces them to work.
Every day, after work, the kidnapped people can walk in the park and talk with each other. However, when three members of three different parties start talking with each other, the wizard reconverts them to the fourth party (there are no conversations with 4 or more people involved).

a) Find out whether it is possible that, after some time, all of the kidnapped people belong to the same party. If the answer is yes, determine to which party they will belong.
b) Find all quartets of positive integers that add up to 101 that if they were to be considered the number of members from the four parties, it is possible that, after some time, all of the kidnapped people belong to the same party, under the same rules imposed by the wizard.

3 Let $ABCD$ be a trapezoid with bases $BC \parallel AD$, where $AD > BC$, and non-parallel legs AB and CD. Let M be the intersection of AC and BD. Let Γ_1 be a circumference that passes through M and is tangent to AD at point A; let Γ_2 be a circumference that passes through M and is tangent to AD at point D. Let S be the intersection of the lines AB and CD, X the intersection of Γ_1 with the line AS, Y the intesection of Γ_2 with the line DS, and O the circumcenter of triangle ASD.
Show that $SO \perp XY$.

Day 2

4 Determine all positive integers n such that the number $n(n+2)(n+4)$ has at most 15 positive divisors.

5 At least 3 players take part in a tennis tournament. Each participant plays exactly one match against each other participant. After the tournament has ended, we find out that each player has won at least one match. (There are no ties in tennis).
Show that in the tournament, there was at least one trio of players A, B, C such that A beat B, B beat C, and C beat A.

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6 Each square of 1×1, of a $n \times n$ grid is colored using red or blue, in such way that between all the 2×2 subgrids, there are all the possible colorations of a 2×2 grid using red or blue, (colorations that can be obtained by using rotation or symmetry, are said to be different, so there are 16 possibilities). Find:
a) The minimum value of n.
b) For that value, find the least possible number of red squares.
