Junior Balkan Team Selection Test 2009

www.artofproblemsolving.com/community/c3850

by Bugi

Day 1

1 Given are natural numbers a, b and n such that $a^2 + 2nb^2$ is a complete square. Prove that the number $a^2 + nb^2$ can be written as a sum of squares of 2 natural numbers.

2 In isosceles right triangle ABC a circle is inscribed. Let CD be the hypotenuse height ($D \in AB$), and let P be the intersection of inscribed circle and height CD. In which ratio does the circle divide segment AP?

3 On each field of the board $n \times n$ there is one figure, where $n \geq 2$. In one move we move every figure on one of its diagonally adjacent fields. After one move on one field there can be more than one figure. Find the least number of fields on which there can be all figures after some number of moves.

4 In the decimal expression of a 2009-digit natural number there are only the digits 5 and 8. Prove that we can get a 2008-digit number divisible by 11 if we remove just one digit from the number.

Day 2

1 Find all two digit numbers $\overline{AB}$ such that $\overline{AB}$ divides $\overline{A0B}$.

2 From the set $\{1, 2, 3, \ldots, 2009\}$ we choose 1005 numbers, such that sum of any 2 numbers isn't neither 2009 nor 2010. Find all ways on we can choose these 1005 numbers.

3 Let $ABCD$ be a convex quadrilateral, such that
$\angle CBD = 2 \cdot \angle ADB, \angle ABD = 2 \cdot \angle CDB$ and $AB = CB$.

Prove that quadrilateral $ABCD$ is a kite.

4 For positive real numbers x, y, z the inequality

$$\frac{1}{x^2+1} + \frac{1}{y^2+1} + \frac{1}{z^2+1} = \frac{1}{2}$$

holds. Prove the inequality

$$\frac{1}{x^3+2} + \frac{1}{y^3+2} + \frac{1}{z^3+2} < \frac{1}{3}.$$