**Stanford Mathematics Tournament 2000**

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– Team

– **p1.** You are given a number, and round it to the nearest thousandth, round this result to nearest hundredth, and round this result to the nearest tenth. If the final result is $.7$, what is the smallest number you could have been given? As is customary, $5$'s are always rounded up. Give the answer as a decimal.

**p2.** The price of a gold ring in a certain universe is proportional to the square of its purity and the cube of its diameter. The purity is inversely proportional to the square of the depth of the gold mine and directly proportional to the square of the price, while the diameter is determined so that it is proportional to the cube root of the price and also directly proportional to the depth of the mine. How does the price vary solely in terms of the depth of the gold mine?

**p3.** Find the sum of all integers from $1$ to $1000$ inclusive which contain at least one $7$ in their digits, i.e. find $7 + 17 + ... + 979 + 987 + 997$.

**p4.** All arrangements of letters $VNNWHTAAIE$ are listed in lexicographic (dictionary) order. If $AAEHINNTVW$ is the first entry, what entry number is $VANNAWHITE$?

**p5.** Given $\cos(a + b) + \sin(a - b) = 0$, $\tan b = \frac{1}{2000}$ , find $\tan a$.

**p6.** If $a$ is a root of $x^3 - x - 1 = 0$, compute the value of

$$a^{10} + 2a^8 - a^7 - 3a^6 - 3a^5 + 4a^4 + 2a^3 - 4a^4 - 6a - 17.$$

**p7.** $8712$ is an integral multiple of its reversal, $2178$, as $8712 = 4 * 2178$. Find another $4$-digit number which is a non-trivial integral multiple of its reversal.

**p8.** A woman has $\$1.58$ in pennies, nickels, dimes, quarters, half-dollars and silver dollars. If she has a different number of coins of each denomination, how many coins does she have?

**p9.** Find all positive primes of the form $4x^4 + 1$, for x an integer.

**p10.** How many times per day do at least two of the three hands on a clock coincide?

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**p11.** Find all polynomials $f(x)$ with integer coefficients such that the coefficients of both $f(x)$ and $[f(x)]^3$ lie in the set $\{0, 1, -1\}$.

**p12.** At a dance, Abhinav starts from point $(a, 0)$ and moves along the negative $x$ direction with speed $v_a$, while Pei-Hsin starts from $(0, b)$ and glides in the negative $y$-direction with speed $v_b$. What is the distance of closest approach between the two?

**p13.** Let $P_1P_2...P_n$ be a convex $n$-gon. If all lines $P_iP_j$ are joined, what is the maximum possible number of intersections in terms of $n$ obtained from strictly inside the polygon?

**p14.** Define a sequence $< x_n >$ of real numbers by specifying an initial $x_0$ and by the recurrence $x_{n+1} = \frac{1+x_n}{10x_n}$ . Find $x_n$ as a function of $x_0$ and $n$, in closed form. There may be multiple cases.

**p15.** $\lim_{n\to\infty} nr\sqrt[2]{1-\cos\frac{2\pi}{n}} =$ ?

PS. You had better use hide for answers.

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**1** If $a = 2b + c$, $b = 2c + d$, $2c = d + a - 1$, $d = a - c$, what is $b$?

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**2** The temperatures $f$°F and $c$°C are equal when $f = \frac{9}{5}c + 32$. What temperature is the same in both °F and °C?

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**3** A twelve foot tree casts a five foot shadow. How long is Henry's shadow (at the same time of day) if he is five and a half feet tall?

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**4** Tickets for the football game are \$10 for students and \$15 for non-students. If 3000 fans attend and pay \$36250, how many students went?

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**5** Find the interior angle between two sides of a regular octagon (degrees).

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**6** Three cards, only one of which is an ace, are placed face down on a table. You select one, but do not look at it. The dealer turns over one of the other cards, which is not the ace (if neither are, he picks one of them randomly to turn over). You get a chance to change your choice and pick either of the remaining two face-down cards. If you selected the cards so as to maximize the chance of finding the ace on the second try, what is the probability that you selected it on the

(a) first try?

(b) second try?

**7** Find $[\sqrt{19992000}]$ where $[x]$ is the greatest integer less than or equal to $x$.

**8** Bobo the clown was juggling his spherical cows again when he realized that when he drops a cow is related to how many cows he started off juggling. If he juggles $1$, he drops it after $64$ seconds. When juggling $2$, he drops one after $55$ seconds, and the other $55$ seconds later. In fact, he was able to create the following table:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| cows started juggling | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| seconds he drops after | 64 | 55 | 47 | 40 | 33 | 27 | 22 | 18 | 1 |
| cows started juggling | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 2 |
| seconds he drops after | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 |

can only juggle up to $22$ cows. To juggle the cows the longest, what number of cows should he start off juggling? How long (in minutes) can he juggle for?

**9** Edward's formula for the stock market predicts correctly that the price of HMMT is directly proportional to a secret quantity $x$ and inversely proportional to $y$, the number of hours he slept the night before. If the price of HMMT is $\$12$ when $x = 8$ and $y = 4$, how many dollars does it cost when $x = 4$ and $y = 8$?

**10** Bob has a $12$ foot by $20$ foot garden. He wants to put fencing around it to keep out the neighbor's dog. Normal fence posts cost $\$2$ each while strong ones cost $\$3$ each. If he needs one fence post for every $2$ feet and has $\$70$ to spend on the fence posts, what is the largest number of strong fence posts he can buy?

**11** If $a@b = \frac{a+b}{a-b}$, find $n$ such that $3@n = 3$.

**13** How many permutations of $123456$ have exactly one number in the correct place?

**14** The author of this question was born on April 24, 1977. What day of the week was that?

**15** Which is greater: $(3^5)^{(5^3)}$ or $(5^3)^{(3^5)}$?

**16** Joe bikes $x$ miles East at $20$ mph to his friend's house. He then turns South and bikes $x$ miles at $20$ mph to the store. Then, Joe turns East again and goes to his grandma's house at $14$ mph. On this last leg, he has to carry flour he bought for her at the store. Her house is $2$ more miles from the store than Joe's friend's house is from the store. Joe spends a total of 1 hour on the bike to get to his grandma's house. If Joe then rides straight home in his grandma's helicopter at $78$ mph, how many minutes does it take Joe to get home from his grandma's house

**17** In how many distinct ways can the letters of STANTON be arranged?

**18** You use a lock with four dials, each of which is set to a number between 0 and 9 (inclusive). You can never remember your code, so normally you just leave the lock with each dial one higher

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than the correct value. Unfortunately, last night someone changed all the values to 5. All you remember about your code is that none of the digits are prime, 0, or 1, and that the average value of the digits is 5.

How many combinations will you have to try?

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**19** Eleven pirates find a treasure chest. When they split up the coins in it, they find that there are 5 coins left. They throw one pirate overboard and split the coins again, only to find that there are 3 coins left over. So, they throw another pirate over and try again. This time, the coins split evenly.

What is the least number of coins there could have been?

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**22** An equilateral triangle with sides of length $4$ has an isosceles triangle with the same base and half the height cut out of it.

Find the remaining area

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**23** What are the last two digits of $7^{7^{7^7}}$?

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**24** Peter is randomly filling boxes with candy. If he has 10 pieces of candy and 5 boxes in a row labeled A, B, C, D, and E, how many ways can he distribute the candy so that no two adjacent boxes are empty?

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**25** How many points does one have to place on a unit square to guarantee that two of them are strictly less than 1/2 unit apart?

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