Stanford Mathematics Tournament 2002

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by augustin_p, parmenides51, Mrdavid445

– Team

– **p1.** Evaluate $\sqrt{i}$ in the form $a + bi$ with $a > 0$, where a and b are real numbers and i is $\sqrt{-1}$.

p2. Let A be the matrix $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & 4 \\ 0 & 6 & 3 \end{bmatrix}$ .What is $det(A^{-1})$?

p3. How many positive integers divide the number of positive integers that divide 2002^{2002}?

p4. In base 6, how many $(2n + 1)$-digit numbers are palindromes?

p5. In the diagram below, what is the sum of five angles θ_i numbered, $\sum_{n=i}^5 \theta_n$?
https://cdn.artofproblemsolving.com/attachments/5/2/21d2302852680db9d2c8b31d3f01c9d0b67b5
png

p6. Let x be the smallest number such that x written out in English (i.c. $1,647$ is one thousand six hundred forty seven) has exactly 300 letters. What is the most common digit $(0 - 9)$ in x?

p7. Define $g(x) = \int_x^{x+1} 2^t dt$, and $g'(x) = \frac{d}{dx} g(x)$. Compute $g'(10)$.

p8. If $xy = 24$ with x and y real, what is the minimum value that $x^2 + 4y^2$ can attain?

p9. Find the cubic polynomial $f(x)$ such that $f(1) = 1$, $f(2) = 5$, $f(3) = 14$, and $f(4) = 30$.

p10. What is $1^2 + 2^2 + 3^2 + ... + 2002^2$?

p11. The r-th power mean P_r of n numbers $x_1, ..., x_n$ is defined as

$$P_r(x_1, ..., x_n) = \left(\frac{x_1^r + ... + x_n^r}{n}\right)^{1/r}$$

for $r \neq 0$, and $P_0 = (x_1x_2...x_n)^{1/n}$ The Power Mean Inequality says that if $r > s$, then $P_r \geq P_s$.

Using this fact, find out how many ordered pairs of positive integers (x, y) satisfy $48\sqrt{xy} - x^2 - y^2 \geq 289$.

p12. After meeting him in the afterlife, Gauss challenges Fermat to a boxing match. Each mathematician is wearing glasses, and Gauss has a $1/3$ probability of knocking off Fermat's glasses during the match, whereas Fermat has a $1/5$ chance of knocking off Gauss's glasses. Each mathematician has a $1/2$ chance of losing without his glasses and a $1/5$ chance of losing anyway with his. Note that it is possible for both Fermat and Gauss to lose (simultaneous knockout) or for neither to lose (the match is a draw). Given that Gauss wins the match (and Fermat loses), what is the probability that Gauss has lost his glasses?

p13. Evaluate

$$\frac{1}{-1+\frac{1}{-1+\frac{1}{-1+...}}}$$

p14. What is the smallest positive integer x such that $x^2 + x + 41$ is not prime?

p15. Let $A(t)$ be an $n \times n$ square matrix whose entries are all functions of t, and suppose that $detA(t) \neq 0$ for all t. Then $\frac{dA}{dt} = A'$ is simply the matrix formed by differentiating each entry of $A(t)$ with respect to t. Write $\frac{d}{dt}(A{-}1(t))$ in terms of $A(t)$ and A', where the only differentiation occurs in A' itself.

PS. You had better use hide for answers .

– Algebra

1 Completely factor the polynomial $x^4 - x^3 - 5x^2 + 3x + 6$

2 Solve for all real x that satisfy the equation $4^x = 2^x + 6$

3 A clockmaker wants to design a clock such that the area swept by each hand (second, minute, and hour) in one minute is the same (all hands move continuously). What is the length of the hour hand divided by the length of the second hand?

4 Suppose that $n^2 - 2m^2 = m(n+3) - 3$. Find all integers m such that all corresponding solutions for n will *not* be real.

5 Solve for a, b, c given that $a \leq b \leq c$, and

$a + b + c = -1$

$ab + bc + ac = -4$

$abc = -2$

6 How many integers x, from 10 to 99 inclusive, have the property that the remainder of x^2 divided by 100 is equal to the square of the units digit of x?

– Geometry

2 Upon cutting a certain rectangle in half, you obtain two rectangles that are scaled down versions of the original. What is the ratio of the longer side length to the shorter side length?

3 An equilateral triangle has has sides 1 inch long. An ant walks around the triangle maintaining a distance of 1 inch from the triangle at all times. How far does the ant walk?