Stanford Mathematics Tournament 2006

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– General

– February 25th

1 After a cyclist has gone $\frac{2}{3}$ of his route, he gets a flat tire. Finishing on foot, he spends twice as long walking as he did riding. How many times as fast does he ride as walk?

2 A customer enters a supermarket. The probability that the customer buys bread is .60, the probability that the customer buys milk is .50, and the probability that the customer buys both bread and milk is .30. What is the probability that the customer would buy either bread or milk or both?

3 After a typist has written ten letters and had addressed the ten corresponding envelopes, a careless mailing clerk inserted the letters in the envelopes at random, one letter per envelope. What is the probability that **exactly** nine letters were inserted in the proper envelopes?

4 In a certain tournament bracket, a player must be defeated three times to be eliminated. If 512 contestants enter the tournament, what is the greatest number of games that could be played?

5 A geometric series is one where the ratio between each two consecutive terms is constant (ex. 3,6,12,24,...). The fifth term of a geometric series is 5!, and the sixth term is 6!. What is the fourth term?

6 An alarm clock runs 4 minutes slow every hour. It was set right $3\frac{1}{2}$ hours ago. Now another clock which is correct shows noon. In how many minutes, to the nearest minute, will the alarm clock show noon?

7 An aircraft is equipped with three engines that operate independently. The probability of an engine failure is .01. What is the probability of a successful flight if only one engine is needed for the successful operation of the aircraft?

8 Given two 2's, "plus" can be changed to "times" without changing the result: 2+2=22. The solution with three numbers is easy too: 1+2+3=123. There are three answers for the five-number case. Which five numbers with this property has the largest sum?

9 If to the numerator and denominator of the fraction $\frac{1}{3}$ you add its denominator 3, the fraction will double. Find a fraction which will triple when its denominator is added to its numerator

and to its denominator and find one that will quadruple.

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10 What is the square root of the sum of the first 2006 positive odd integers?

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11 An insurance company believes that people can be divided into 2 classes: those who are accident prone and those who are not. Their statistics show that an accident prone person will have an accident in a yearly period with probability 0.4, whereas this probability is 0.2 for the other kind. Given that 30

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12 What is the largest prime factor of 8091?

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13 123456789=100. Here is the only way to insert 7 pluses and/or minus signs between the digits on the left side to make the equation correct: 1+2+3-4+5+6+78+9=100. Do this with only three plus or minus signs.

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14 Determine the area of the region defined by *x*+*y* and *y* sin *x*.

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15 The odometer of a family car shows 15,951 miles. The driver noticed that this number is palindromic: it reads the same backward as forwards. "Curious," the driver said to himself, "it will be a long time before that happens again." Surprised, he saw his third palindromic odometer reading (not counting 15,951) exactly five hours later. How many miles per hour was the car traveling in those 5 hours (assuming speed was constant)?

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16 Points A_1, A_2, ... are placed on a circle with center O such that $\angle OA_nA_{n+1} = 35^\circ$ and $A_n \neq A_{n+2}$ for all positive integers n. What is the smallest $n > 1$ for which $A_n = A_1$?

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17 Car A is traveling at 20 miles per hour. Car B is 1 mile behind, following at 30 miles per hour. A fast fly can move at 40 miles per hour. The fly begins on the front bumper of car B, and flies back and forth between the two cars. How many miles will the fly travel before it is crushed in the collision?

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18 Alex and Brian take turns shooting free throws until they each shoot twice. Alex and Brian have 80

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19 When the celebrated German mathematician Karl Gauss (1777-1855) was nine years old, he was asked to add all the integers from 1 through 100. He quickly added 1 and 100, 2 and 99, and so on for 50 pairs of numbers each adding in 101. His answer was 50 101=5,050. Now find the sum of all the digits in the integers from 1 through 1,000,000 (i.e. all the digits in those numbers, not the numbers themselves).

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20 Given a random string of 33 bits (0 or 1), how many (they can overlap) occurrences of two consecutive 0's would you expect? (i.e. "100101" has 1 occurrence, "0001" has 2 occurrences)

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21 How many positive integers less than 2005 are relatively prime to 1001?

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22 A certain college student had the night of February 23 to work on a chemistry problem set and a math problem set (both due on February 24, 2006). If the student worked on his problem sets in the math library, the probability of him finishing his math problem set that night is 95

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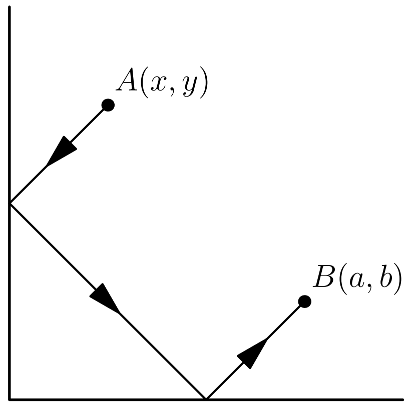
23 Consider two mirrors placed at a right angle to each other and two points A at (x, y) and B at (a, b). Suppose a person standing at point A shines a laser pointer so that it hits both mirrors and then hits a person standing at point B (as shown in the picture). What is the total distance that the light ray travels, in terms of a, b, x, and y? Assume that x, y, a, and b are positive.



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24 The number 555,555,555,555 factors into eight distinct prime factors, each with a multiplicity of 1. What are the three largest prime factors of 555,555,555,555?

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25 For positive integers n let $D(n)$ denote the set of positive integers that divide n and let $S(n) = \Sigma_{k\in D(n)}\frac{1}{k}$. What is $S(2006)$? Answer with a fraction reduced to lowest terms.

– Algebra

– February 25th

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1 A finite sequence of positive integers m_i for $i = 1, 2, ..., 2006$ are defined so that $m_1 = 1$ and $m_i = 10m_{i-1} + 1$ for $i > 1$. How many of these integers are divisible by 37?

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2 Find the minimum value of $2x^2 + 2y^2 + 5z^2 - 2xy - 4yz - 4x - 2z + 15$ for real numbers x, y, z.

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3 A Gaussian prime is a Gaussian integer $z = a+bi$ (where a and b are integers) with no Gaussian integer factors of smaller absolute value. Factor $-4 + 7i$ into Gaussian primes with positive real parts. i is a symbol with the property that $i^2 = -1$.

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4 Simplify: $\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-a)(b-c)} + \frac{c^3}{(c-a)(c-b)}$

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5 Jerry is bored one day, so he makes an array of Cocoa pebbles. He makes 8 equal rows with the pebbles remaining in a box. When Kramer drops by and eats one, Jerry yells at him until Kramer realizes he can make 9 equal rows with the remaining pebbles. After Kramer eats another, he finds he can make 10 equal rows with the remaining pebbles. Find the smallest number of pebbles that were in the box in the beginning.

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6 Let a, b, c be real numbers satisfying:

$$ab - a = b + 119$$

$$bc - b = c + 59$$

$$ca - c = a + 71$$

Determine all possible values of $a + b + c$.

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7 Find all solutions to $aabb = n^4 - 6n^3$, where a and b are non-zero digits, and n is an integer. (a and b are not necessarily distinct.)

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8 Evaluate:

$$\sum_{x=2}^{10} \frac{2}{x(x^2-1)}$$

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9 Principal Skinner is thinking of two integers m and n and bets Superintendent Chalmers that he will not be able to determine these integers with a single piece of information. Chalmers asks Skinner the numerical value of $mn+13m+13n-m^2-n^2$. From the value of this expression alone, he miraculously determines both m and n. What is the value of the above expression.

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10 Evaluate: $\sum_{k=1}^{\infty} \frac{k}{a^{k-1}}$ for all $|a| < 1$.

– Team

– February 25th

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1 Given $\triangle ABC$, where A is at $(0,0)$, B is at $(20,0)$, and C is on the positive y-axis. Cone M is formed when $\triangle ABC$ is rotated about the x-axis, and cone N is formed when $\triangle ABC$ is rotated about the y-axis. If the volume of cone M minus the volume of cone N is 140π, find the length of $\overline{BC}$.

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2 In a given sequence $\{S_1, S_2, ..., S_k\}$, for terms $n \geq 3$, $S_n = \sum_{i=1}^{n-1} i \cdot S_{n-i}$. For example, if the first two elements are 2 and 3, respectively, the third entry would be $1 \cdot 3 + 2 \cdot 2 = 7$, and the fourth would be $1 \cdot 7 + 2 \cdot 3 + 3 \cdot 2 = 19$, and so on. Given that a sequence of integers having this form starts with 2, and the 7th element is 68, what is the second element?

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3 A triangle has altitudes of length 5 and 7. What is the maximum length of the third altitude?

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4 Let $x + y = a$ and $xy = b$. The expression $x^6 + y^6$ can be written as a polynomial in terms of a and b. What is this polynomial?

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5 There exist two positive numbers x such that $\sin(\arccos(\tan(\arcsin x))) = x$. Find the product of the two possible x.

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6 The expression $16^n + 4^n + 1$ is equiavalent to the expression $(2^{p(n)} - 1)/(2^{q(n)} - 1)$ for all positive integers $n > 1$ where $p(n)$ and $q(n)$ are functions and $\frac{p(n)}{q(n)}$ is constant. Find $p(2006) - q(2006)$.

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7 Let S be the set of all 3-tuples (a, b, c) that satisfy $a + b + c = 3000$ and $a, b, c > 0$. If one of these 3-tuples is chosen at random, what's the probability that a, b or c is greater than or equal to 2,500?

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8 Evaluate: $\lim_{n\to\infty} \sum_{k=n^2}^{(n+1)^2} \dfrac{1}{\sqrt{k}}$

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9 $\triangle ABC$ has $AB = AC$. Points M and N are midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. The medians $\overline{MC}$ and $\overline{NB}$ intersect at a right angle. Find $(\frac{AB}{BC})^2$.

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10 Find the smallest positive m for which there are at least 11 even and 11 odd positive integers n so that $\frac{n^3+m}{n+2}$ is an integer.

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11 Polynomial $P(x) = c_{2006}x^{2006} + c_{2005}x^{2005} + \ldots + c_1x + c_0$ has roots $r_1, r_2, \ldots, r_{2006}$. The coefficients satisfy $2i\frac{c_i}{c_{2006-i}} = 2j\frac{c_j}{c_{2006-j}}$ for all pairs of integers $0 \leq i, j \leq 2006$. Given that $\sum_{i\neq j, i=1, j=1}^{2006} \frac{r_i}{r_j} = 42$, determine $\sum_{i=1}^{2006}(r_1 + r_2 + \ldots + r_{2006})$.

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12 Find the total number of k-tuples $(n_1, n_2, ..., n_k)$ of positive integers so that $n_{i+1} \geq n_i$ for each i, and k regular polygons with numbers of sides $n_1, n_2, ..., n_k$ respectively will fit into a tesselation at a point. That is, the sum of one interior angle from each of the polygons is 360°.

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13 A ray is drawn from the origin tangent to the graph of the upper part of the hyperbola $y^2 = x^2 - x + 1$ in the first quadrant. This ray makes an angle of θ with the positive x-axis. Compute $\cos\theta$.

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14 Find the smallest nonnegative integer n for which $\binom{2006}{n}$ is divisible by 7^3.

15 Let c_i denote the ith composite integer so that $\{c_i\} = 4, 6, 8, 9, ...$ Compute

$$\prod_{i=1}^{\infty} \frac{c_i^2}{c_i^2 - 1}$$

(Hint: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$)

– Advanced Topics

– February 25th

1 A college student is about to break up with her boyfriend, a mathematics major who is apparently more interested in math than her. Frustrated, she cries, You mathematicians have no soul! Its all numbers and equations! What is the root of your incompetence?! Her boyfriend assumes she means the square root of himself, or the square root of i. What two answers should he give?

3 Simplify: $\sum_{k=10}^{2006} \binom{k}{10}$ (Your answer should contain no summations but may still contain binomial coefficients/combinations.)

4 Rice University and Stanford University write questions and corresponding solutions for a high school math tournament. The Rice group writes 10 questions every hour but make a mistake in calculating their solutions 10

5 Evaluate: $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k+2}+(k+2)\sqrt{k}}$

6 Ten teams of five runners each compete in a cross-country race. A runner finishing in *n*th place contributes *n* points to his team, and there are no ties. The team with the lowest score wins. Assuming the first place team does not have the same score as any other team, how many winning scores are possible?

7 A lattice point in the plane is a point whose coordinates are both integers. Given a set of 100 distinct lattice points in the plane, find the smallest number of line segments $\overline{AB}$ for which A and B are distinct lattice points in this set and the midpoint of $\overline{AB}$ is also a lattice point (not necessarily in the set).

10 Evaluate: $\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2-n+1}\right)$