Stanford Mathematics Tournament 2007

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–	Team
1	How many rational solutions for x are there to the equation $x^4+(2-p)x^3+(2-2p)x^2+(1-2p)x-p=0$ if p is a prime number?
2	If a and b are each randomly and independently chosen in the interval $[-1,1]$, what is the probability that $|a|+|b|<1$?
3	A clock currently shows the time $10:10$. The obtuse angle between the hands measures x degrees. What is the next time that the angle between the hands will be x degrees? Round your answer to the nearest minute.
4	What is the area of the smallest triangle with all side lengths rational and all vertices lattice points?
5	How many five-letter "words" can you spell using the letters S, I, and T, if a "word" is defines as any sequence of letters that does not contain three consecutive consonants?
6	$x\equiv\left(\sum_{k=1}^{2007}k\right)\mod 2016$, where $0\le x\le 2015$. Solve for x.
7	Daniel counts the number of ways he can form a positive integer using the digits $1,2,2,3$, and 4 (each digit at most once). Edward counts the number of ways you can use the letters in the word "$BANANAS$" to form a six-letter word (it doesn't have to be a real English word). Fernando counts the number of ways you can distribute nine identical pieces of candy to three children. By their powers combined, they add each of their values to form the number that represents the meaning of life. What is the meaning of life? (Hint: it's not 42.)
8	A 13-foot tall extraterrestrial is standing on a very small spherical planet with radius 156 feet. It sees an ant crawling along the horizon. If the ant circles the extraterrestrial once, always staying on the horizon, how far will it travel (in feet)?
9	Let d_n denote the number of derangements of the integers $1,2,\dots n$ so that no integer i is in the i^{th} position. It is possible to write a recurrence relation $d_n=f(n)d_{n-1}+g(n)d_{n-2}$; what is $f(n)+g(n)$?
10	A nondegenerate rhombus has side length l, and its area is twice that of its inscribed circle. Find the radius of the inscribed circle.

11 The polynomial $R(x)$ is the remainder upon dividing x^{2007} by x^2-5x+6. $R(0)$ can be expressed as $ab(a^c-b^c)$. Find $a+c-b$.

12 Brownian motion (for example, pollen grains in water randomly pushed by collisions from water molecules) simplified to one dimension and beginning at the origin has several interesting properties. If $B(t)$ denotes the position of the particle at time t, the average of $B(t)$ is $x=0$, but the averate of $B(t)^2$ is t, and these properties of course still hold if we move the space and time origins ($x=0$ and $t=0$) to a later position and time of the particle (past and future are independent). What is the average of the product $B(t)B(s)$?

13 Mary Jane and Rachel are playing ping pong. Rachel has a $7/8$ chance of returning any shot, and Mary Jane has a $5/8$ chance. Mary Jane serves to Rachel (and doesn't mess up the serve). What is the average number of returns made?

14 Let p, q be positive integers and let $x_0=0$. Suppose $x_{n+1}=x_n+p+\sqrt{q^2+4px_n}$. Find an explicit formula for x_n.

15 Evaluate $\int_0^\infty \frac{\tan^{-1}(\pi x)-\tan^{-1}x}{x}dx$

– General

– March 4th

1 There are three bins: one with 30 apples, one with 30 oranges, and one with 15 of each. Each is labeled "apples," "oranges," or "mixed." Given that all three labels are wrong, how many pieces of fruit must you look at to determine the correct labels?

2 Aliens from Lumix have one head and four legs, while those from Obscra have two heads and only one leg. If 60 aliens attend a joint Lumix and Obscra interworld conference, and there are 129 legs present, how many heads are there?

3 Mary puts one red and one blue marble into a box. In another box she places two red marbles. She then forgets which box is which and randomly reaches into one of the boxes and takes out a red marble. What is the probability that the other marble in that box is blue?

4 Evaluate $(\tan 10^\circ)(\tan 20^\circ)(\tan 30^\circ)(\tan 40^\circ)(\tan 50^\circ)(\tan 60^\circ)(\tan 70^\circ)(\tan 80^\circ)$.

5 Two disks of radius 1 are drawn so that each disk's circumference passes through the center of the other disk. What is the circumference of the region in which they overlap?

6 Team Stanford has a $\frac{1}{3}$ chance of winning any given math contest. If Stanford competes in 4 contests this quarter, what is the probability that the team will win at least once?

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7 A boat is traveling upstream at 5 mph relative to the current flowing against it at 1 mph. If a tree branch 10 miles upstream from the boat falls into the current of the river, how many hours does it take to reach the boat?

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8 Tina writes four letters to her friends Silas, Jessica, Katie, and Lekan. She prepares an envelope for Silas, an envelope for Jessica, an envelope for Katie, and an envelope for Lekan. However, she puts each letter into a random envelope. What is the probability that no one receives the letter they are supposed to receive?

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9 Peter Pan and Crocodile are each getting hired for a job. Peter wants to get paid 6.4 dollars daily, but Crocodile demands to be paid 10 cents on day 1, 20 cents on day 2, 40 cents on day 3, 80 cents on day 4, and so on. After how many whole days will Crocodile's total earnings exceed that of Peter's?

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10 Al, Betty, and Clara are in the same class of 50 students total, but are not friends with each other. Al is friends with 24 students, Betty is friends with 39, and Clara is friends with 20. What is the greatest number of students that could be friends with all 3 of them?

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11 Jonathan finds out that his ideal match is Sara Lark, but to improve his odds of finding a girlfriend, he is willing to date any girl whose name is an anagram of "Sara Lark," and whose name consists of both a first and last name of at least one letter. How many such anagrams are there?

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12 Pete has some trouble slicing a 20-inch (diameter) pizza. His first two cuts (from center to circumference of the pizza) make a 30º slice. He continues making cuts until he has gone around the whole pizza, each time trying to copy the angle of the previous slice but in fact adding 2º each time. That is, he makes adjacent slices of 30º, 32º, 34º, and so on. What is the area of the smallest slice?

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13 A rope of length 10 *m* is tied tautly from the top of a flagpole to the ground 6 *m* away from the base of the pole. An ant crawls up the rope and its shadow moves at a rate of 30 *cm/min*. How many meters above the ground is the ant after 5 minutes? (This takes place on the summer solstice on the Tropic of Cancer so that the sun is directly overhead.)

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14 Let there be 50 natural numbers a_i such that $0 < a_1 < a_2 < ... < a_{50} < 150$. What is the greatest possible sum of the differences d_j where each $d_j = a_{j+1} - a_j$?

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15 A number x is uniformly chosen on the interval $[0, 1]$, and y is uniformly randomly chosen on $[-1, 1]$. Find the probability that $x > y$.

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16 Find the area of a square inscribed in an equilateral triangle, with one edge of the square on an edge of the triangle, if the side length of the triangle is $2+\sqrt{3}$.

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17 There is a test for the dangerous bifurcation virus that is 99% accurate. In other words, if someone has the virus, there is a 99% chance that the test will be positive, and if someone does not have it, then there is a 99% chance the test will be negative. Assume that exactly 1% of the general population has the virus. Given an individual that has tested positive from this test, what is the probability that he or she actually has the disease? Express your answer as a percentage.

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18 A farmer wants to build a rectangular region, using a river as one side and some fencing as the other three sides. He has 1200 feet of fence which he can arrange to different dimensions. He creates the rectangular region with length L and width W to enclose the greatest area. Find $L+W$.

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19 Arrange the following four numbers from smallest to largest $a=(10^{100})^{10}$, $b=10^{(10^{10})}$, $c=1000000!$, $d=(100!)^{10}$

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20 Let there be $4n+2$ distinct paths in space with exactly $2n^2+6n+1$ points at which exactly two of the paths intersect. (A path never intersects itself.) What is the maximum number of points where exactly three paths intersect?

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21 Convert the following decimal to a common fraction in lowest terms: $0.920072007200720072007\ldots$ (or $0.9\overline{2007}$).

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22 Katie begins juggling five balls. After every second elapses, there is a chance she will drop a ball. If she is currently juggling k balls, this probability is $\frac{k}{10}$. Find the expected number of seconds until she has dropped all the balls.

– Algebra

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1 Find all real roots of f if $f(x^{1/9})=x^2-3x-4$.

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2 Given that $x_1>0$ and $x_2=4x_1$ are solutions to ax^2+bx+c and that $3a=2(c-b)$, what is x_1?

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3 Let a,b,c be the roots of $x^3-7x^2-6x+5=0$. Compute $(a+b)(a+c)(b+c)$.

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4 How many positive integers n, with $n\le 2007$, yield a solution for x (where x is real) in the equation $\lfloor x\rfloor+\lfloor 2x\rfloor+\lfloor 3x\rfloor=n$?

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5 The polynomial $-400x^5+2660x^4-3602x^3+1510x^2+18x-90$ has five rational roots. Suppose you guess a rational number which could possibly be a root (according to the rational root theorem). What is the probability that it actually is a root?

6	What is the largest prime factor of $4^9 + 9^4$?
7	Find the minimum value of $xy + x + y + \frac{1}{xy} + \frac{1}{x} + \frac{1}{y}$ for $x, y > 0$ real.
8	If $r + s + t = 3$, $r^2 + s^2 + t^2 = 1$, and $r^3 + s^3 + t^3 = 3$, compute rst.
9	Find $a^2 + b^2$ given that a, b are real and satisfy $$a = b + \frac{1}{a + \frac{1}{b + \frac{1}{a + \cdots}}}; b = a - \frac{1}{b + \frac{1}{a - \frac{1}{b + \cdots}}}$$
10	Evaluate $$\sum_{k=1}^{2007} (-1)^k k^2$$