Stanford Mathematics Tournament 2008

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– Team

– **p1**. Find the maximum value of $e^{\sin x \cos x \tan x}$.

p2. A fighter pilot finds that the average number of enemy ZIG planes she shoots down is $56z - 4z^2$, where z is the number of missiles she fires. Intending to maximize the number of planes she shoots down, she orders her gunner to ”Have a nap ... then fire z missiles!” where z is an integer. What should z be?

p3. A sequence is generated as follows: if the n^{th} term is even, then the $(n + 1)^{th}$ term is half the n^{th} term; otherwise it is two more than twice the n^{th} term. If the first term is 10, what is the 2008^{th} term?

p4. Find the volume of the solid formed by rotating the area under the graph of $y = \sqrt{x}$ around the x-axis, with $0 \le x \le 2$.

p5. Find the volume of a regular octahedron whose vertices are at the centers of the faces of a unit cube.

p6. What is the area of the triangle with vertices $(x, 0, 0)$, $(0, y, 0)$, and $(0, 0, z)$?

p7. Daphne is in a maze of tunnels shown below. She enters at A, and at each intersection, chooses a direction randomly (including possibly turning around). Once Daphne reaches an exit, she will not return into the tunnels. What is the probability that she will exit at A?
`https://cdn.artofproblemsolving.com/attachments/c/0/0f8777e9ac9cbe302f042d040e8864d68cadb`
`png`

p8. In triangle AXE, T is the midpoint of $\overline{EX}$, and P is the midpoint of $\overline{ET}$. If triangle APE is equilateral, find $\cos(m\angle XAE)$.

p9. In rectangle $XKCD$, J lies on $\overline{KC}$ and Z lies on $\overline{XK}$. If $\overline{XJ}$ and $\overline{KD}$ intersect at Q, $\overline{QZ} \perp \overline{XK}$, and $\frac{KC}{KJ} = n$, find $\frac{XD}{QZ}$.

p10. Bill the magician has cards A, B, and C as shown. For his act, he asks a volunteer to pick any number from 1 through 8 and tell him which cards among A, B, and C contain it. He then uses this information to guess the volunteer's number (for example, if the volunteer told Bill "A and C", he would guess "3").
One day, Bill loses card C and cannot remember which numbers were on it. He is in a hurry and randomly chooses four different numbers from 1 to 8 to write on a card. What is the probability Bill will still be able to do his trick?

A: 2 3 5 7

B: 2 4 6 7

C: 2 3 6 1

p11. Given that $f(x, y) = x^7y^8 + x^4y^{14} + A$ has root $(16, 7)$, find another root.

p12. How many nonrectangular trapezoids can be formed from the vertices of a regular octagon?

p13. If $re^{i\theta}$ is a root of $x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0$, $r > 0$, and $0 \le \theta < 360$ with θ in degrees, find all possible values of θ.

p14. For what real values of n is $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\tan(x))^n dx$ defined?

p15. A parametric graph is given by

$$\begin{cases} y = \sin t \\ x = \cos t + \frac{1}{2}t \end{cases}$$

How many times does the graph intersect itself between $x = 1$ and $x = 40$?

PS. You had better use hide for answers.

–	February 23rd
1	Calculate the least integer greater than $5^{(-6)(-5)(-4)\ldots(2)(3)(4)}$.
2	How many primes exist which are less than 50?
3	Give the positive root(s) of $x^3 + 2x^2 - 2x - 4$.

4 A right triangle has sides of integer length. One side has length 11. What is the area of the triangle?

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5 One day, the temperature increases steadily from a low of $45°$F in the early morning to a high of $70°$F in the late afternoon. At how many times from early morning to late afternoon was the temperature an integer in both Fahrenheit and Celsius? Recall that $C = \frac{5}{9}(F - 32)$.

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6 A round pencil has length 8 when unsharpened, and diameter $\frac{1}{4}$. It is sharpened perfectly so that it remains 8 inches long, with a 7 inch section still cylindrical and the remaining 1 inch giving a conical tip. What is its volume?

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7 At the Rice Mathematics Tournament, 80

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8 Terence Tao is playing rock-paper-scissors. Because his mental energy is focused on solving the twin primes conjecture, he uses the following very simple strategy:

·He plays rock first.
·On each subsequent turn, he plays a different move than the previous one, each with probability ½.

What is the probability that his 5th move will be rock?

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9 What is the sum of the prime factors of 20!?

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10 Six people play the following game: They have a cube, initially white. One by one, the players mark an X on a white face of the cube, and roll it like a die. The winner is the first person to roll an X (for example, player 1 wins with probability $\frac{1}{6}$, while if none of players 1-5 win, player 6 will place an X on the last square and win for sure). What is the probability that the sixth player wins?

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11 Simplify: $\sqrt[3]{\frac{17\sqrt{7}+45}{4}}$

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13 Let N be the number of distinct rearrangements of the 34 letters in SUPERCALIFRAGILISTICEXPIALIDOCIOUS. How many positive factors does N have?

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14 Suppose families always have one, two, or three children, with probability ¼, ½, ¼ respectively. Assuming everyone eventually gets married and has children, what is the probability of a couple having exactly four grandchildren?

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15 While out for a stroll, you encounter a vicious velociraptor. You start running away to the northeast at 10m/s, and you manage a three-second head start over the raptor. If the raptor runs at

$15\sqrt{2}$m/s, but only runs either north or east at any given time, how many seconds do you have until it devours you?

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16 Suppose convex hexagon HEXAGN has $120°$-rotational symmetry about a point P—that is, if you rotate it $120°$ about P, it doesn't change. If $PX = 1$, find the area of triangle $\triangle GHX$.
