Stanford Mathematics Tournament 2012

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– Team

– **p1.** How many functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ take on exactly 3 distinct values?

p2. Let i be one of the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$. Suppose that for all positive integers n, the number n^n never has remainder i upon division by 12. List all possible values of i.

p3. A card is an ordered 4-tuple (a_1, a_2, a_3, a_4) where each a_i is chosen from $\{0, 1, 2\}$. A line is an (unordered) set of three (distinct) cards $\{(a_1, a_2, a_3, a_4)$,(b_1, b_2, b_3, b_4),$(c_1, c_2, c_3, c_4)\}$ such that for each i, the numbers a_i, b_i, c_i are either all the same or all different. How many different lines are there?

p4. We say that the pair of positive integers (x, y), where $x < y$, is a k-tangent pair if we have $\arctan \frac{1}{k} = arctan\frac{1}{x} + arctan\frac{1}{y}$. Compute the second largest integer that appears in a 2012-tangent pair.

p5. Regular hexagon $A_1A_2A_3A_4A_5A_6$ has side length 1. For $i = 1, ..., 6$, choose B_i to be a point on the segment A_iA_{i+1} uniformly at random, assuming the convention that $A_{j+6} = A_j$ for all integers j. What is the expected value of the area of hexagon $B_1B_2B_3B_4B_5B_6$?

p6. Evaluate $\sum^{\infty}_{n=1} \sum^{\infty}_{m=1} \frac{1}{nm(n+m+1)}$.

p7. A plane in 3-dimensional space passes through the point (a_1, a_2, a_3), with a_1, a_2, and a_3 all positive. The plane also intersects all three coordinate axes with intercepts greater than zero (i.e. there exist positive numbers b_1, b_2, b_3 such that $(b_1, 0, 0)$, $(0, b_2, 0)$, and $(0, 0, b_3)$ all lie on this plane). Find, in terms of a_1, a_2, a_3, the minimum possible volume of the tetrahedron formed by the origin and these three intercepts.

p8. The left end of a rubber band e meters long is attached to a wall and a slightly sadistic child holds on to the right end. A point-sized ant is located at the left end of the rubber band at time $t = 0$, when it begins walking to the right along the rubber band as the child begins stretching it. The increasingly tired ant walks at a rate of $1/(ln(t + e))$ centimeters per second, while the child uniformly stretches the rubber band at a rate of one meter per second. The rubber band is infinitely stretchable and the ant and child are immortal. Compute the time in seconds, if it exists, at which the ant reaches the right end of the rubber band. If the ant never reaches the right end, answer $+\infty$.

p9. We say that two lattice points are neighboring if the distance between them is 1. We say that a point lies at distance d from a line segment if d is the minimum distance between the point and any point on the line segment. Finally, we say that a lattice point A is nearby a line segment if the distance between A and the line segment is no greater than the distance between the line segment and any neighbor of A. Find the number of lattice points that are nearby the line segment connecting the origin and the point $(1984, 2012)$.

p10. A permutation of the first n positive integers is valid if, for all $i > 1$, i comes after $\lfloor \frac{i}{2} \rfloor$ in the permutation. What is the probability that a random permutation of the first 14 integers is valid?

p11. Given that $x, y, z > 0$ and $xyz = 1$, find the range of all possible values of $\frac{x^3+y^3+z^3-x^{-3}-y^{-3}-z^{-3}}{x+y+z-x^{-1}-y^{-1}-z^{-1}}$.

p12. A triangle has sides of length $\sqrt2$, $3+\sqrt3$, and $2\sqrt2+\sqrt6$. Compute the area of the smallest regular polygon that has three vertices coinciding with the vertices of the given triangle.

p13. How many positive integers n are there such that for any natural numbers a, b, we have $n|(a^2b + 1)$ implies $n|(a^2 + b)$?

p14. Find constants a and c such that the following limit is finite and nonzero: $c = \lim_{n\to\infty} \frac{e\left(1-\frac{1}{n}\right)^n-1}{n^a}$. Give your answer in the form (a, c).

p15. Sean thinks packing is hard, so he decides to do math instead. He has a rectangular sheet that he wants to fold so that it fits in a given rectangular box. He is curious to know what the optimal size of a rectangular sheet is so that it's expected to fit well in any given box. Let a and b be positive reals with $a \le b$, and let m and n be independently and uniformly distributed random variables in the interval $(0, a)$. For the ordered 4-tuple (a, b, m, n), let $f(a, b, m, n)$ denote the ratio between the area of a sheet with dimension a×b and the area of the horizontal cross-section of the box with dimension $m \times n$ after the sheet has been folded in halves along each dimension until it occupies the largest possible area that will still fit in the box (because Sean is picky, the sheet must be placed with sides parallel to the box's sides). Compute the smallest value of b/a that maximizes the expectation f.

PS. You had better use hide for answers.

– Algebra

1 Compute the minimum possible value of

$(x-1)^2+(x-2)^2+(x-3)^2+(x-4)^2+(x-5)^2$

For real values x

2 Find all real values of x such that $(\frac{1}{5}(x^2-10x+26))^{x^2-6x+5}=1$

3 Express $\frac{2^3-1}{2^3+1}\times\frac{3^3-1}{3^3+1}\times\frac{4^3-1}{4^3+1}\times\cdots\times\frac{16^3-1}{16^3+1}$ as a fraction in lowest terms.

4 If x, y, and z are integers satisfying $xyz+4(x+y+z)=2(xy+xz+yz)+7$, list all possibilities for the ordered triple (x,y,z).

5 The quartic (4th-degree) polynomial P(x) satisfies $P(1)=0$ and attains its maximum value of 3 at both $x=2$ and $x=3$. Compute $P(5)$.

6 There exist two triples of real numbers (a,b,c) such that $a-\frac{1}{b}, b-\frac{1}{c}, c-\frac{1}{a}$ are the roots to the cubic equation $x^3-5x^2-15x+3$ listed in increasing order. Denote those (a_1,b_1,c_1) and (a_2,b_2,c_2). If a_1, b_1, and c_1 are the roots to monic cubic polynomial f and a_2,b_2, and c_2 are the roots to monic cubic polynomial g, find $f(0)^3+g(0)^3$

8 For real numbers (x,y,z) satisfying the following equations, find all possible values of $x+y+z$

$x^2y+y^2z+z^2x=-1$

$xy^2+yz^2+zx^2=5$

$xyz=-2$

9 Find the minimum value of xy, given that $x^2+y^2+z^2=7$, $xz+xy+yz=4$, and x,y,z are real numbers

10 Let X_1, X_2, ..., X_{2012} be chosen independently and uniformly at random from the interval $(0,1]$. In other words, for each X_n, the probability that it is in the interval $(a,b]$ is $b-a$. Compute the probability that $\lceil\log_2 X_1\rceil+\lceil\log_4 X_2\rceil+\cdots+\lceil\log_{1024} X_{2012}\rceil$ is even. (Note: For any real number a, $\lceil a\rceil$ is defined as the smallest integer not less than a.)

– Geometry

3 Let ABC be an equilateral triangle of side 1. Draw three circles O_a, O_b, O_c with diameters BC, CA, and AB, respectively. Let S_a denote the area of the region inside O_a and outside of O_b and O_c. Define S_b and S_c similarly, and let S be the area of intersection between the three circles. Find $S_a+S_b+S_c-S$.

–	Advanced Topics
1	Define a number to be $boring$ if all the digits of the number are the same. How many positive integers less than 10000 are both prime and boring?
2	Find the sum of all integers x, $x \geq 3$, such that 201020112012_x (that is, 201020112012 interpreted as a base x number) is divisible by $x-1$
3	Given that $\log_{10} 2 \approx 0.30103$, find the smallest positive integer n such that the decimal representation of 2^{10n} does not begin with the digit 1.
4	Two different squares are randomly chosen from an 8×8 chessboard. What is the probability that two queens placed on the two squares can attack each other? Recall that queens in chess can attack any square in a straight line vertically, horizontally, or diagonally from their current position.