Stanford Mathematics Tournament 2013

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–	General
1	Robin goes birdwatching one day. he sees three types of birds: penguins, pigeons, and robins. $\frac{2}{3}$ of the birds he sees are robins. $\frac{1}{8}$ of the birds he sees are penguins. He sees exactly 5 pigeons. How many robins does Robin see?
2	Jimmy runs a successful pizza shop. In the middle of a busy day, he realizes that he is running low on ingredients. Each pizza must have 1 lb of dough, $\frac{1}{4}$ lb of cheese, $\frac{1}{6}$ lb of sauce, and $\frac{1}{3}$ lb of toppings, which include pepperonis, mushrooms, olives, and sausages. Given that Jimmy currently has 200 lbs of dough, 20 lbs of cheese, 20 lbs of sauce, 15 lbs of pepperonis, 5 lbs of mushrooms, 5 lbs of olives, and 10 lbs of sausages, what is the maximum number of pizzas that JImmy can make?
3	Queen Jack likes a 5-card hand if and only if the hand contains only queens and jacks. Considering all possible 5-card hands that can come from a standard 52-card deck, how many hands does Queen Jack like?
4	What is the smallest number over 9000 that is divisible by the first four primes?
5	A rhombus has area 36 and the longer diagonal is twice as long as the shorter diagonal. What is the perimeter of the rhombus?
6	Nick is a runner, and his goal is to complete four laps around a circuit at an average speed of 10 mph. If he completes the first three laps at a constant speed of only 9 mph, what speed does he need to maintain in miles per hour on the fourth lap to achieve his goal?
7	A fly and an ant are on one corner of a unit cube. They wish to head to the opposite corner of the cube. The fly can fly through the interior of the cube, while the ant has to walk across the faces of the cube. How much shorter is the fly's path if both insects take the shortest path possible?
8	According to Moor's Law, the number of shoes in Moor's room doubles every year. In 2013, Moor's room starts out having exactly one pair of shoes. If shoes always come in unique, matching pairs, what is the earliest year when Moor has the ability to wear at least 500 mismatches pairs of shoes? Note that left and right shoes are distinct, and Moor must always wear one of each.

9 A tree has 10 pounds of apples at dawn. Every afternoon, a bird comes and eats x pounds of apples. Overnight, the amount of food on the tree increases by 10

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10 Consider a sequence given by $a_n = a_{n-1} + 3a_{n-2} + a_{n-3}$, where $a_0 = a_1 = a_2 = 1$. What is the remainder of a_{2013} divided by 7?

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11 Sara has an ice cream cone with every meal. The cone has a height of $2\sqrt{2}$ inches and the base of the cone has a diameter of 2 inches. Ice cream protrudes from the top of the cone in a perfect hempisphere. Find the surface area of the ice cream cone, ice cream included, in square inches.

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12 What is the greatest possible value of c such that $x^2+5x+c = 0$ has at least one real solution?

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13 $\mathbb{R}^2$-tic-tac-toe is a game where two players take turns putting red and blue points anywhere on the xy plane. The red player moves first. The first player to get 3 of their points in a line without any of their opponent's points in between wins. What is the least number of moves in which Red can guarantee a win? (We count each time that Red places a point as a move, including when Red places its winning point.)

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14 Peter is chasing after Rob. Rob is running on the line $y = 2x+5$ at a speed of 2 units a second, starting at the point $(0, 5)$. Peter starts running t seconds after Rob, running at 3 units a second. Peter also starts at $(0, 5)$ and catches up to Rob at the point $(17, 39)$. What is the value of t?

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15 Given regular hexagon $ABCDEF$, compute the probability that a randomly chosen point inside the hexagon is inside triangle PQR, where P is the midpoint of AB, Q is the midpoint of CD, and R is the midpoint of EF.

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16 Eight people are posing together in a straight line for a photo. Alice and Bob must stand next to each other, and Claire and Derek must stand next to each other. How many different ways can the eight people pose for their photo?

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17 An isosceles right triangle is inscribed in a circle of radius 5, thereby separating the circle into four regions. Compute the sum of the areas of the two smallest regions.

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18 Caroline wants to plant 10 trees in her orchard. Planting n apple trees requires n^2 square meters, planting n apricot trees requires $5n$ square meters, and planting n plum trees requires n^3 square meters. If she is committed to growing only apple, apricot, and plum trees, what is the least amount of space in square meters, that her garden will take up?

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19 A triangle with side lengths 2 and 3 has an area of 3. Compute the third side length of the triangle.

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20 Ben is throwing darts at a circular target with diameter 10. Ben never misses the target when he throws a dart, but he is equally likely to hit any point on the target. Ben gets $\lceil 5-x \rceil$ points for having the dart land x units away from the center of the target. What is the expected number of points that Ben can earn from throwing a single dart? (Note that $\lceil y \rceil$ denotes the smallest integer greater than or equal to y.)

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21 How many positive three-digit integers $\underline{abc}$ can represent a valid date in 2013, where either a corresponds to a month and $\underline{bc}$ corresponds to the day in that month, or $\underline{ab}$ corresponds to a month and c corresponds to the day? For example, 202 is a valid representation for February 2nd, and 121 could represent either January 21st or December 1st.

(Note: During the actual test they had to write the number of days in each month so don't feel bad if you have to google that :P)

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22 The set $A = \{1, 2, 3, \cdots, 10\}$ contains the numbers 1 through 10. A subset of A of size n is competent if it contains n as an element. A subset of A is minimally competent if it itself is competent, but none of its proper subsets are. Find the total number of minimally competent subsets of A.

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23 Let a and b be the solutions to $x^2 - 7x + 17 = 0$. Compute $a^4 + b^4$.

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24 Compute the square of the distance between the incenter (center of the inscribed circle) and circumcenter (center of the circumscribed circle) of a 30-60-90 right triangle with hypotenuse of length 2.

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25 A 3×6 grid is filled with the numbers in the list $\{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9\}$ according to the following rules: (1) Both the first three columns and the last three columns contain the integers 1 through 9. (2) No numbers appear more than once in a given row. Let N be the number of ways to fill the grid and let k be the largest positive integer such that 2^k divides N. What is k?

– Calculus

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2 Compute all real values of b such that, for $f(x) = x^2 + bx - 17$, $f(4) = f'(4)$.

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3 Suppose a and b are real numbers such that

$$\lim_{x \to 0} \frac{\sin^2 x}{e^{ax} - bx - 1} = \frac{1}{2}.$$

Determine all possible ordered pairs (a, b).

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4 Evaluate $\int_0^4 e^{\sqrt{x}}\, dx$.

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6 Compute $\sum_{k=0}^{\infty} \int_0^{\frac{\pi}{3}} \sin^{2k} x \, dx$.

7 The function $f(x)$ has the property that, for some real positive constant C, the expression

$$\frac{f^{(n)}(x)}{n+x+C}$$

is independent of n for all nonnegative integers n, provided that $n+x+C \neq 0$. Given that $f'(0)=1$ and $\int_0^1 f(x)\,dx = C + (e-2)$, determine the value of C.
Note: $f^{(n)}(x)$ is the n-th derivative of $f(x)$, and $f^{(0)}(x)$ is defined to be $f(x)$.

8 The function $f(x)$ is defined for all $x \geq 0$ and is always nonnegative. It has the additional property that if any line is drawn from the origin with any positive slope m, it intersects the graph $y=f(x)$ at precisely one point, which is $\frac{1}{\sqrt{m}}$ units from the origin. Let a be the unique real number for which f takes on its maximum value at $x=a$ (you may assume that such an a exists). Find $\int_0^a f(x)\,dx$.

9 Evaluate $\int_0^{\pi/2} \frac{dx}{\left(\sqrt{\sin x}+\sqrt{\cos x}\right)^4}$.

10 Evaluate $\lim_{n\to\infty} \left[\left(\prod_{k=1}^{n} \frac{2k}{2k-1}\right) \int_{-1}^{\infty} \frac{(\cos x)^{2n}}{2^x}\,dx\right]$.

– Advanced Topics Tiebreaker

1 Andrew flips a fair coin 5 times, and counts the number of heads that appear. Beth flips a fair coin 6 times and also counts the number of heads that appear. Compute the probability Andrew counts at least as many heads as Beth.

2 How many alphabetic sequences (that is, sequences containing only letters from $a \cdots z$) of length 2013 have letters in alphabetic order?

3 Suppose two equally strong tennis players play against each other until one player wins three games in a row. The results of each game are independent, and each player will win with probability $\frac{1}{2}$. What is the expected value of the number of games they will play?

– Algebra Tiebreakers

1 x is a base-10 number such that when the digits of x are interpreted as a base-20 number, the resulting number is twice the value as when they are interpreted as a base-13 number. Find the sum of all possible values of x.

2 If f is a monic cubic polynomial with $f(0) = -64$, and all roots of f are non-negative real numbers, what is the largest possible value of $f(-1)$? (A polynomial is monic if it has a leading coefficient of 1.)

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3 Find the minimum of $f(x, y, z) = x^3 + 12\frac{yz}{x} + 16(\frac{1}{yz})^{\frac{3}{2}}$ where x, y, and z are all positive.
Note: The problem as given in the tiebreaker did not specify that each of x, y, and z had to be positive. Without this constraint, the answer is $-\infty$, as x^3 can be an arbitrarily large negative value and dominate the expression.

– Geometry Tiebreakers

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1 A circle of radius 2 is inscribed in equilateral triangle ABC. The altitude from A to BC intersects the circle at a point D not on BC. BD intersects the circle at a point E distinct from D. Find the length of BE.

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2 Points A, B, and C lie on a circle of radius 5 such that $AB = 6$ and $AC = 8$. Find the smaller of the two possible values of BC.

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3 In quadrilateral $ABCD$, diagonals AC and BD intersect at E. If $AB = BE = 5$, $EC = CD = 7$, and $BC = 11$, compute AE.

– Geometry

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1 In triangle ABC, $AC = 7$. D lies on AB such that $AD = BD = CD = 5$. Find BC.

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2 What is the perimeter of a rectangle of area 32 inscribed in a circle of radius 4?

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3 Robin has obtained a circular pizza with radius 2. However, being rebellious, instead of slicing the pizza radially, he decides to slice the pizza into 4 strips of equal width both vertically and horizontally. What is the area of the smallest piece of pizza?

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4 $ABCD$ is a regular tetrahedron with side length 1. Find the area of the cross section of $ABCD$ cut by the plane that passes through the midpoints of AB, AC, and CD.

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5 In square $ABCD$ with side length 2, let P and Q both be on side AB such that $AP = BQ = \frac{1}{2}$. Let E be a point on the edge of the square that maximizes the angle PEQ. Find the area of triangle PEQ.

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6 $ABCD$ is a rectangle with $AB = CD = 2$. A circle centered at O is tangent to BC, CD, and AD (and hence has radius 1). Another circle, centered at P, is tangent to circle O at point T and is also tangent to AB and BC. If line AT is tangent to both circles at T, find the radius of circle P.

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7 $ABCD$ is a square such that AB lies on the line $y = x + 4$ and points C and D lie on the graph of parabola $y^2 = x$. Compute the sum of all possible areas of $ABCD$.

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8 Let equilateral triangle ABC with side length 6 be inscribed in a circle and let P be on arc AC such that $AP \cdot PC = 10$. Find the length of BP.

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9 In tetrahedron $ABCD$, $AB = 4, CD = 7$, and $AC = AD = BC = BD = 5$. Let I_A, I_B, I_C, and I_D denote the incenters of the faces opposite vertices A, B, C, and D, respectively. It is provable that AI_A intersects BI_B at a point X, and CI_C intersects DI_D at a point Y. Compute XY.

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10 Let triangle ABC have side lengths $AB = 16, BC = 20, AC = 26$. Let $ACDE, ABFG$, and $BCHI$ be squares that are entirely outside of triangle ABC. Let J be the midpoint of EH, K be the midpoint of DG, and L be the midpoint of AC. Find the area of triangle JKL.

– Algebra

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1 Nick is a runner, and his goal is to complete four laps around a circuit at an average speed of 10 mph. If he completes the first three laps at a constant speed of only 9 mph, what speed does he need to maintain in miles per hour on the fourth lap to achieve his goal?

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2 A tree has 10 pounds of apples at dawn. Every afternoon, a bird comes and eats x pounds of apples. Overnight, the amount of food on the tree increases by 10%. What is the maximum value of x such that the bird can sustain itself indefinitely on the tree without the tree running out of food?

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3 Karl likes the number 17 his favorite polynomials are monic quadratics with integer coefficients such that 17 is a root of the quadratic and the roots differ by no more than 17. Compute the sum of the coefficients of all of Karl's favorite polynomials. (A monic quadratic is a quadratic polynomial whose x^2 term has a coefficient of 1.)

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4 Given that $f(x) + 2f(8 - x) = x^2$ for all real x, compute $f(2)$.

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5 For exactly two real values of b, b_1 and b_2, the line $y = bx - 17$ intersects the parabola $y = x^2 + 2x + 3$ at exactly one point. Compute $b_1^2 + b_2^2$.

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6 Compute the largest root of $x^4 - x^3 - 5x^2 + 2x + 6$.

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7 Find all real x that satisfy $\sqrt[3]{20x + \sqrt[3]{20x + 13}} = 13$.

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8 Find the sum of all real x such that

$$\frac{4x^2 + 15x + 17}{x^2 + 4x + 12} = \frac{5x^2 + 16x + 18}{2x^2 + 5x + 13}.$$

9 Let $a = -\sqrt{3} + \sqrt{5} + \sqrt{7}, b = \sqrt{3} - \sqrt{5} + \sqrt{7}, c = \sqrt{3} + \sqrt{5} - \sqrt{7}$. Evaluate

$$\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)}.$$

10 Given a complex number z such that $z^{13} = 1$, find all possible value of $z+z^3+z^4+z^9+z^{10}+z^{12}$.

– Team

1 Let $f_1(n)$ be the number of divisors that n has, and define $f_k(n) = f_1(f_{k-1}(n))$. Compute the smallest integer k such that $f_k(2013^{2013}) = 2$.

2 In unit square $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. Let M be the midpoint of $\overline{CD}$, with $\overline{AM}$ intersecting $\overline{BD}$ at F and $\overline{BM}$ intersecting $\overline{AC}$ at G. Find the area of quadrilateral $MFEG$.

3 Nine people are practicing the triangle dance, which is a dance that requires a group of three people. During each round of practice, the nine people split off into three groups of three people each, and each group practices independently. Two rounds of practice are different if there exists some person who does not dance with the same pair in both rounds. How many different rounds of practice can take place?

4 For some positive integers a and b, $(x^a + abx^{a-1} + 13)^b(x^3 + 3bx^2 + 37)^a = x^{42} + 126x^{41} + \cdots$. Find the ordered pair (a, b).

5 A polygonal prism is made from a flexible material such that the two bases are regular 2^n-gons $(n > 1)$ of the same size. The prism is bent to join the two bases together without twisting, giving a figure with 2^n faces. The prism is then repeatedly twisted so that each edge of one base becomes aligned with each edge of the base exactly once. For an arbitrary n, what is the sum of the number of faces over all of these configurations (including the non-twisted case)?

6 How many distinct sets of 5 distinct positive integers A satisfy the property that for any positive integer $x \leq 29$, a subset of A sums to x?

7 Find all real values of u such that the curves $y = x^2 + u$ and $y = \sqrt{x-u}$ intersect in exactly one point.

8 Rational Man and Irrational Man both buy new cars, and they decide to drive around two racetracks from time $t = 0$ to time $t = \infty$. Rational Man drives along the path parametrized by

$$x = \cos(t)$$
$$y = \sin(t)$$

and Irrational Man drives along the path parametrized by

$$x = 1 + 4\cos\frac{t}{\sqrt{2}}$$
$$y = 2\sin\frac{t}{\sqrt{2}}.$$

Find the largest real number d such that at any time t, the distance between Rational Man and Irrational Man is not less than d.

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9 Charles is playing a variant of Sudoku. To each lattice point (x, y) where $1 \le x, y < 100$, he assigns an integer between 1 and 100 inclusive. These integers satisfy the property that in any row where $y = k$, the 99 values are distinct and never equal to k; similarly for any column where $x = k$. Now, Charles randomly selects one of his lattice points with probability proportional to the integer value he assigned to it. Compute the expected value of $x + y$ for the chosen point (x, y).

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10 A unit circle is centered at the origin and a tangent line to the circle is constructed in the first quadrant such that it makes an angle $5\pi/6$ with the y-axis. A series of circles centered on the x-axis are constructed such that each circle is both tangent to the previous circle and the original tangent line. Find the total area of the series of circles.

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11 What is the smalles positive integer with exactly 768 divisors? Your answer may be written in its prime factorization.

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12 Suppose Robin and Eddy walk along a circular path with radius r in the same direction. Robin makes a revolution around the circular path every 3 minutes and Eddy makes a revolution every minute. Jack stands still at a distance $R > r$ from the center of the circular path. At time $t = 0$, Robin and Eddy are at the same point on the path, and Jack, Robin, and Eddy, and the center of the path are collinear. When is the next time the three people (but not necessarily the center of the path) are collinear?

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13 A board has 2, 4, and 6 written on it. A person repeatedly selects (not necessarily distinct) values for x, y, and z from the board, and writes down $xyz + xy + yz + zx + x + y + z$ if and only if that number is not yet on the board and is also less than or equal to 2013. This person repeats this process until no more numbers can be written. How many numbers will be written at the end of the process?

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14 You have a 2 meter long string. You choose a point along the string uniformly at random and make a cut. You discard the shorter section. If you still have 0.5 meters or more of string, you repeat. You stop once you have less than 0.5 meters of string. On average, how many cuts will you make before stopping?

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15 Suppose we climb a mountain that is a cone with radius 100 and height 4. We start at the bottom of the mountain (on the perimeter of the base of the cone), and our destination is the opposite side of the mountain, halfway up (height $z = 2$). Our climbing speed starts at $v_0 = 2$ but gets slower at a rate inversely proportional to the distance to the mountain top (so at height z the speed v is $(h - z)v_0/h$). Find the minimum time needed to get to the destination.

– Advanced Topics

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1 How many positive three-digit integers $\underline{a}\,\underline{b}\,\underline{c}$ can represent a valid date in 2013, where either a corresponds to a month and $\underline{b}\,\underline{c}$ corresponds to the day in that month, or $\underline{a}\,\underline{b}$ corresponds to a month and c corresponds to the day? For example, 202 is a valid representation for February 2nd, and 121 could represent either January 21st or December 1st.

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2 Consider the numbers $\{24, 27, 55, 64, x\}$. Given that the mean of these five numbers is prime and the median is a multiple of 3, compute the sum of all possible positive integral values of x.

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3 Nick has a terrible sleep schedule. He randomly picks a time between 4 AM and 6 AM to fall asleep, and wakes up at a random time between 11 AM and 1 PM of the same day. What is the probability that Nick gets between 6 and 7 hours of sleep?

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4 Given the digits 1 through 7, one can form $7! = 5040$ numbers by forming different permutations of the 7 digits (for example, 1234567 and 6321475 are two such permutations). If the 5040 numbers are then placed in ascending order, what is the 2013^{th} number?

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5 An unfair coin lands heads with probability $\frac{1}{17}$ and tails with probability $\frac{16}{17}$. Matt flips the coin repeatedly until he flips at least one head and at least one tail. What is the expected number of times that Matt flips the coin?

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6 A positive integer $b \geq 2$ is *neat* if and only if there exist positive base-b digits x and y (that is, x and y are integers, $0 < x < b$ and $0 < y < b$) such that the number $x.y$ base b (that is, $x + \frac{y}{b}$) is an integer multiple of x/y. Find the number of *neat* integers less than or equal to 100.

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7 Robin is playing notes on an 88-key piano. He starts by playing middle C, which is actually the 40th lowest note on the piano (i.e. there are 39 notes lower than middle C). After playing a note, Robin plays with probability $\frac{1}{2}$ the lowest note that is higher than the note he just played, and with probability $\frac{1}{2}$ the highest note that is lower than the note he just played. What is the probability that he plays the highest note on the piano before playing the lowest note?

8 Farmer John owns 2013 cows. Some cows are enemies of each other, and Farmer John wishes to divide them into as few groups as possible such that each cow has at most 3 enemies in her group. Each cow has at most 61 enemies. Compute the smallest integer G such that, no matter which enemies they have, the cows can always be divided into at most G such groups?

9 Big candles cost 16 cents and burn for exactly 16 minutes. Small candles cost 7 cents and burn for exactly 7 minutes. The candles burn at possibly varying and unknown rates, so it is impossible to predictably modify the amount of time for which a candle will burn except by burning it down for a known amount of time. Candles may be arbitrarily and instantly put out and relit. Compute the cost in cents of the cheapest set of big and small candles you need to measure exactly 1 minute.

10 Compute the number of positive integers b where $b \le 2013$, $b \neq 17$, and $b \neq 18$, such that there exists some positive integer N such that $\frac{N}{17}$ is a perfect 17th power, $\frac{N}{18}$ is a perfect 18th power, and $\frac{N}{b}$ is a perfect bth power.