European Mathematical Cup 2016

www.artofproblemsolving.com/community/c386575

by steppewolf

– Senior Division

---

1 Is there a sequence $a_1, ..., a_{2016}$ of positive integers, such that every sum

$$a_r + a_{r+1} + ... + a_{s-1} + a_s$$

(with $1 \leq r \leq s \leq 2016$) is a composite number, but:
a) $GCD(a_i, a_{i+1}) = 1$ for all $i = 1, 2, ..., 2015$;
b) $GCD(a_i, a_{i+1}) = 1$ for all $i = 1, 2, ..., 2015$ and $GCD(a_i, a_{i+2}) = 1$ for all $i = 1, 2, ..., 2014$?
$GCD(x, y)$ denotes the greatest common divisor of x, y.

Proposed by Matija Buci

---

2 For two positive integers a and b, Ivica and Marica play the following game: Given two piles of a
and b cookies, on each turn a player takes $2n$ cookies from one of the piles, of which he eats n and puts n of
them on the other pile. Number n is arbitrary in every move. Players take turns alternatively, with Ivica going
first. The player who cannot make a move, loses. Assuming both players play perfectly, determine all pairs of
numbers (a, b) for which Marica has a winning strategy.

Proposed by Petar Orli

---

3 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that equality

$$f(x + y + yf(x)) = f(x) + f(y) + xf(y)$$

holds for all real numbers x, y.

Proposed by Athanasios Kontogeorgis

---

4 Let C_1, C_2 be circles intersecting in X, Y . Let A, D be points on C_1 and B, C on C_2 such that A, X, C are collinear and D, X, B are collinear. The tangent to circle C_1 at D intersects BC and the tangent to C_2 at B in P, R respectively. The tangent to C_2 at C intersects AD and tangent to C_1 at A, in Q, S respectively. Let W be the intersection of AD with the tangent to C_2 at B and Z the intersection of BC with the tangent to C_1 at A. Prove that the circumcircles of triangles YWZ, RSY and PQY have two points in common, or are tangent in the same point.

Proposed by Misiakos Panagiotis

– Junior Division

1 A grasshopper is jumping along the number line. Initially it is situated at zero. In k-th step, the length of his jump is k.
a) If the jump length is even, then it jumps to the left, otherwise it jumps to the right (for example, firstly
it jumps one step to the right, then two steps to the left, then three steps to the right, then four steps to
the left...). Will it visit on every integer at least once?
b) If the jump length is divisible by three, then it jumps to the left, otherwise it jumps to the right (for
example, firstly it jumps one step to the right, then two steps to the right, then three steps to the left,
then four steps to the right...). Will it visit every integer at least once?

Proposed by Matko Ljulj

2 Two circles C_1 and C_2 intersect at points A and B. Let P, Q be points on circles C_1, C_2 respectively, such that $|AP| = |AQ|$. The segment PQ intersects circles C_1 and C_2 in points M, N respectively. Let C be the center of the arc BP of C_1 which does not contain point A and let D be the center of arc BQ of C_2 which does not contain point A Let E be the intersection of CM and DN. Prove that AE is perpendicular to CD.

Proposed by Steve Dinh

3 Prove that for all positive integers n there exist n distinct, positive rational numbers with sum of
their squares equal to n.

Proposed by Daniyar Aubekerov

4 We will call a pair of positive integers (n, k) with $k > 1$ a $lovely\ couple$ if there exists a table nxn
consisting of ones and zeros with following properties:
In every row there are exactly k ones.
For each two rows there is exactly one column such that on both intersections of that column with the
mentioned rows, number one is written.
Solve the following subproblems:
a) Let $d \neq 1$ be a divisor of n. Determine all remainders that d can give when divided by 6.
b) Prove that there exist infinitely many lovely couples.

Proposed by Miroslav Marinov, Daniel Atanasov

