Math Prize for Girls Problems 2015

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by Ravi B

1 In how many different ways can 900 be expressed as the product of two (possibly equal) positive integers? Regard $m \cdot n$ and $n \cdot m$ as the same product.

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2 Let x and y be real numbers such that

$$2 < \frac{x-y}{x+y} < 5.$$

If $\frac{x}{y}$ is an integer, what is its value?

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3 What is the area of the region bounded by the graphs of $y = |x+2| - |x-2|$ and $y = |x+1| - |x-3|$?

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4 A *binary palindrome* is a positive integer whose standard base 2 (binary) representation is a palindrome (reads the same backward or forward). (Leading zeroes are not permitted in the standard representation.) For example, 2015 is a binary palindrome, because in base 2 it is 11111011111. How many positive integers less than 2015 are binary palindromes?

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5 How many distinct positive integers can be expressed in the form $ABCD - DCBA$, where $ABCD$ and $DCBA$ are 4-digit positive integers? (Here A, B, C and D are digits, possibly equal.) Clarification: A and D can't be zero (because otherwise $ABCD$ or $DCBA$ wouldn't be a true 4-digit integer).

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6 In baseball, a player's *batting average* is the number of hits divided by the number of at bats, rounded to three decimal places. Danielle's batting average is $.399$. What is the fewest number of at bats that Danielle could have?

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7 Let n be a positive integer. In n-dimensional space, consider the 2^n points whose coordinates are all ± 1. Imagine placing an n-dimensional ball of radius 1 centered at each of these 2^n points. Let B_n be the largest n-dimensional ball centered at the origin that does not intersect the interior of any of the original 2^n balls. What is the smallest value of n such that B_n contains a point with a coordinate greater than 2?

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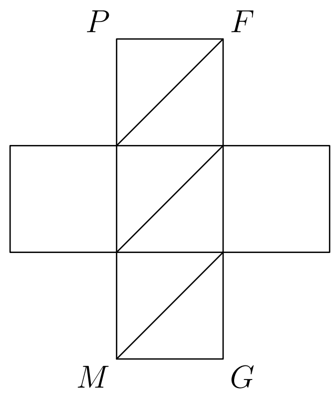
8 In the diagram below, how many different routes are there from point M to point P using only the line segments shown? A route is not allowed to intersect itself, not even at a single point.



9 Say that a rational number is *special* if its decimal expansion is of the form $0.\overline{abcdef}$, where a, b, c, d, e, and f are digits (possibly equal) that include each of the digits 2, 0, 1, and 5 at least once (in some order). How many special rational numbers are there?

10 Among all pairs of real numbers (x, y) such that $\sin\sin x = \sin\sin y$ with $-10\pi \le x, y \le 10\pi$, Oleg randomly selected a pair (X, Y). Compute the probability that $X = Y$.

11 Let $A = (2, 0)$, $B = (0, 2)$, $C = (-2, 0)$, and $D = (0, -2)$. Compute the greatest possible value of the product $PA \cdot PB \cdot PC \cdot PD$, where P is a point on the circle $x^2 + y^2 = 9$.

12 A *permutation* of a finite set is a one-to-one function from the set onto itself. A *cycle* in a permutation P is a nonempty sequence of distinct items $x_1, \ldots, x_n$ such that $P(x_1) = x_2$, $P(x_2) = x_3$, $\ldots$, $P(x_n) = x_1$. Note that we allow the 1-cycle x_1 where $P(x_1) = x_1$ and the 2-cycle x_1, x_2 where $P(x_1) = x_2$ and $P(x_2) = x_1$. Every permutation of a finite set splits the set into a finite number of disjoint cycles. If this number equals 2, then the permutation is called *bi-cyclic*. Compute the number of bi-cyclic permutations of the 7-element set formed by the letters of "PROBLEM".

13 Joel selected an acute angle x (strictly between 0 and 90 degrees) and wrote the values of $\sin x$, $\cos x$, and $\tan x$ on three different cards. Then he gave those cards to three students, Malvina, Paulina, and Georgina, one card to each, and asked them to figure out which trigonometric function (sin, cos, or tan) produced their cards. Even after sharing the values on their cards with each other, only Malvina was able to surely identify which function produced the value on her card. Compute the sum of all possible values that Joel wrote on Malvina's card.

14 Let C be a three-dimensional cube with edge length 1. There are 8 equilateral triangles whose vertices are vertices of C. The 8 planes that contain these 8 equilateral triangles divide C into several nonoverlapping regions. Find the volume of the region that contains the center of C.

15 Let z_1, z_2, z_3, and z_4 be the four distinct complex solutions of the equation

$$z^4 - 6z^2 + 8z + 1 = -4(z^3 - z + 2)i.$$

Find the sum of the six pairwise distances between z_1, z_2, z_3, and z_4.

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16 An ant begins at a vertex of a convex regular icosahedron (a figure with 20 triangular faces and 12 vertices). The ant moves along one edge at a time. Each time the ant reaches a vertex, it randomly chooses to next walk along any of the edges extending from that vertex (including the edge it just arrived from). Find the probability that after walking along exactly six (not necessarily distinct) edges, the ant finds itself at its starting vertex.

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17 Let S be the sum of all distinct real solutions of the equation

$$\sqrt{x + 2015} = x^2 - 2015.$$

Compute $\lfloor 1/S \rfloor$. Recall that if r is a real number, then $\lfloor r \rfloor$ (the *floor* of r) is the greatest integer that is less than or equal to r.

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18 Let n be a positive integer. When the leftmost digit of (the standard base 10 representation of) n is shifted to the rightmost position (the units position), the result is $n/3$. Find the smallest possible value of the sum of the digits of n.

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19 Sabrina has a fair tetrahedral die whose faces are numbered 1, 2, 3, and 4, respectively. She creates a sequence by rolling the die and recording the number on its bottom face. However, she discards (without recording) any roll such that appending its number to the sequence would result in two consecutive terms that sum to 5. Sabrina stops the moment that all four numbers appear in the sequence. Find the expected (average) number of terms in Sabrina's sequence.

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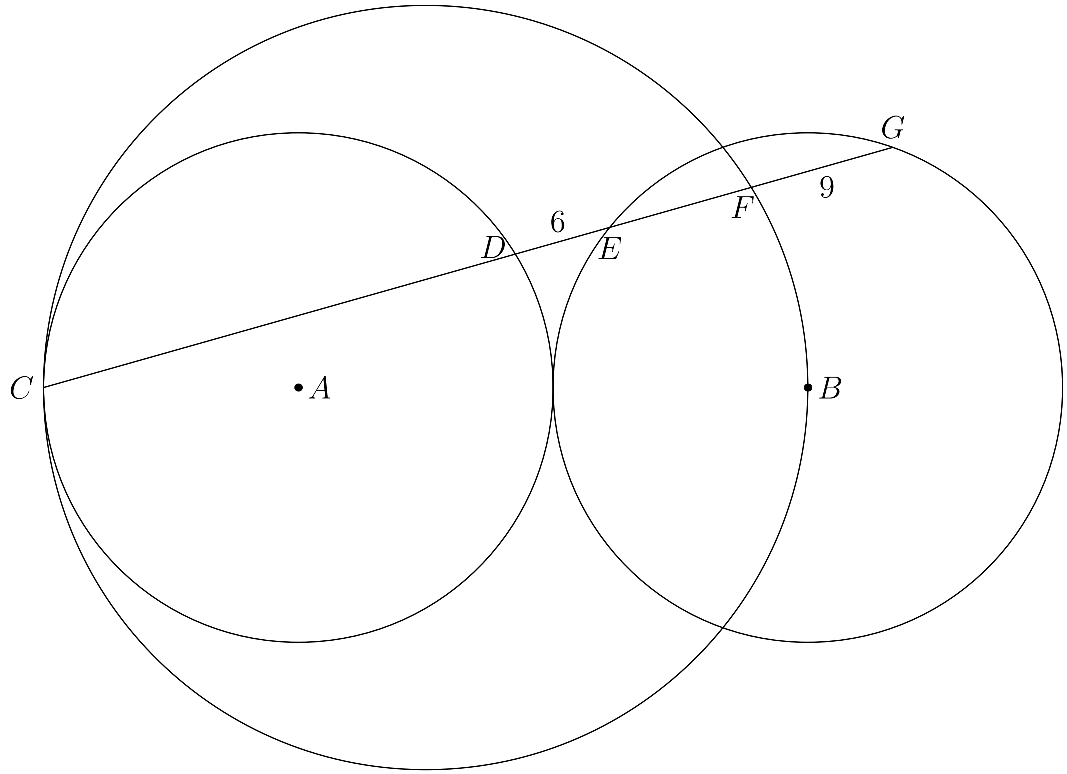
20 In the diagram below, the circle with center A is congruent to and tangent to the circle with center B. A third circle is tangent to the circle with center A at point C and passes through point B. Points C, A, and B are collinear. The line segment $\overline{CDEFG}$ intersects the circles at the indicated points. Suppose that $DE = 6$ and $FG = 9$. Find AG.


G
9
F
6
D
E
C
A
B