Math Prize for Girls Problems 2016

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by Ravi B

1 Let T be a triangle with side lengths 3, 4, and 5. If P is a point in or on T, what is the greatest possible sum of the distances from P to each of the three sides of T?

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2 Katrine has a bag containing 4 buttons with distinct letters M, P, F, G on them (one letter per button). She picks buttons randomly, one at a time, without replacement, until she picks the button with letter G. What is the probability that she has at least three picks and her third pick is the button with letter M?

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3 Compute the least possible value of $ABCD - AB \times CD$, where $ABCD$ is a 4-digit positive integer, and AB and CD are 2-digit positive integers. (Here A, B, C, and D are digits, possibly equal. Neither A nor C can be zero.)

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4 Compute the smallest positive integer n such that 2016^n does not divide $2016!$.

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5 A permutation of a finite set S is a one-to-one function from S to S. A permutation P of the set $\{1, 2, 3, 4, 5\}$ is called a W-permutation if $P(1) > P(2) < P(3) > P(4) < P(5)$. A permutation of the set $\{1, 2, 3, 4, 5\}$ is selected at random. Compute the probability that it is a W-permutation.

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6 The largest term in the binomial expansion of $(1 + \frac{1}{2})^{31}$ is of the form $\frac{a}{b}$, where a and b are relatively prime positive integers. What is the value of b? As an example of a binomial expansion, the binomial expansion of an expression of the form $(x + y)^3$ is the sum of four terms

$$x^3 + 3x^2y + 3xy^2 + y^3.$$

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7 Let S be the set of all real numbers x such that $0 \le x \le 2016\pi$ and $\sin x < 3\sin(x/3)$. The set S is the union of a finite number of disjoint intervals. Compute the total length of all these intervals.

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8 A *strip* is the region between two parallel lines. Let A and B be two strips in a plane. The intersection of strips A and B is a parallelogram P. Let A' be a rotation of A in the plane by 60°. The intersection of strips A' and B is a parallelogram with the same area as P. Let x° be the measure (in degrees) of one interior angle of P. What is the greatest possible value of the number x?

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9 How many distinct lines pass through the point $(0, 2016)$ and intersect the parabola $y = x^2$ at two lattice points? (A lattice point is a point whose coordinates are integers.)

10 How many solutions of the equation $\tan x = \tan \tan x$ are on the interval $0 \le x \le \tan^{-1} 942$? (Here $\tan^{-1}$ means the inverse tangent function, sometimes written $\arctan$.)

11 Compute the number of ordered pairs of complex numbers (u, v) such that $uv = 10$ and such that the real and imaginary parts of u and v are integers.

12 Let b_1, b_2, b_3, c_1, c_2, and c_3 be real numbers such that for every real number x, we have

$$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = (x^2 + b_1 x + c_1)(x^2 + b_2 x + c_2)(x^2 + b_3 x + c_3).$$

Compute $b_1 c_1 + b_2 c_2 + b_3 c_3$.

13 Alice, Beth, Carla, Dana, and Eden play a game in which each of them simultaneously closes her eyes and randomly chooses two of the others to point at (one with each hand). A participant loses if she points at someone who points back at her; otherwise, she wins. Find the probability that all five girls win.

14 We call a set X of real numbers *three-averaging* if for every two distinct elements a and b of X, there exists an element c in X (different from both a and b) such that the number $(a + b + c)/3$ also belongs to X. For instance, the set $\{0, 1008, 2016\}$ is three-averaging. What is the least possible number of elements in a three-averaging set with more than 3 elements?

15 Let H be a convex, equilateral heptagon whose angles measure (in degrees) 168°, 108°, 108°, 168°, x°, y°, and z° in clockwise order. Compute the number y.

16 Let $A < B < C < D$ be positive integers such that every three of them form the side lengths of an obtuse triangle. Compute the least possible value of D.

17 We define the weight W of a positive integer as follows: $W(1) = 0$, $W(2) = 1$, $W(p) = 1 + W(p+1)$ for every odd prime p, $W(c) = 1 + W(d)$ for every composite c, where d is the greatest proper factor of c. Compute the greatest possible weight of a positive integer less than 100.

18 Let $T = \{1, 2, 3, \ldots, 14, 15\}$. Say that a subset S of T is *handy* if the sum of all the elements of S is a multiple of 5. For example, the empty set is handy (because its sum is 0) and T itself is handy (because its sum is 120). Compute the number of handy subsets of T.

19 In the coordinate plane, consider points $A = (0, 0)$, $B = (11, 0)$, and $C = (18, 0)$. Line ℓ_A has slope 1 and passes through A. Line ℓ_B is vertical and passes through B. Line ℓ_C has slope -1 and passes through C. The three lines ℓ_A, ℓ_B, and ℓ_C begin rotating clockwise about points A, B, and C, respectively. They rotate at the same angular rate. At any given time, the three lines form a triangle. Determine the largest possible area of such a triangle.

20 Let a_1, a_2, a_3, a_4, and a_5 be random integers chosen independently and uniformly from the set $\{0, 1, 2, \ldots, 23\}$. (Note that the integers are not necessarily distinct.) Find the probability that

$$\sum_{k=1}^{5} \operatorname{cis}\left(\frac{a_k \pi}{12}\right) = 0.$$

(Here $\operatorname{cis}\theta$ means $\cos\theta + i\sin\theta$.)