

National Math Olympiad (Second Round) 1983www.artofproblemsolving.com/community/c3867

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1 Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be two functions such that $g \circ f : \mathbb{R} \rightarrow \mathbb{R}$ is an injective function. Prove that f is also injective.

2 Prove that the number $x = \sqrt{1 + \sqrt{2}}$ is irrational.

3 Find a matrix $A_{(2 \times 2)}$ for which

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

4 The point M moves such that the sum of squares of the lengths from M to faces of a cube, is fixed. Find the locus of M .

5 Find the value of $S_n = \arctan \frac{1}{2} + \arctan \frac{1}{8} + \arctan \frac{1}{18} + \cdots + \arctan \frac{1}{2n^2}$. Also find $\lim_{n \rightarrow \infty} S_n$.

6 Suppose that

$$f(x) = \begin{cases} n, & n \in \mathbb{N}, x = \frac{1}{n} \\ x, & \text{otherwise} \end{cases}$$

i) In which points, the function has a limit?

ii) Prove that there does not exist limit of f in the point $x = 0$.

7 Find the sum $\sum_{i=1}^{\infty} \frac{n}{2^n}$.
