National Math Olympiad (Second Round) 1984

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1 Let f and g be two functions such that

$$f(x) = \frac{1}{\lfloor |x| \rfloor}, \quad g(x) = \frac{1}{|\lfloor x \rfloor|}.$$

Find the domains of f and g and then prove that

$$\lim_{x \to -1^+} f(x) = \lim_{x \to 1^-} g(x).$$

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2 Consider the function

$$f(x) = \sin\left(\frac{\pi}{2} \lfloor x \rfloor\right).$$

Find the period of f and sketch diagram of f in one period. Also prove that $\lim_{x \to 1} f(x)$ does not exist.

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3 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x+y) = f(x) \cdot f(y) \qquad \forall x, y \in \mathbb{R}$$

Suppose that $f(0) \neq 0$ and $f(0)$ exists and it is finite $(f(0) \neq \infty)$. Prove that f has derivative in each point $x \in \mathbb{R}$.

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4 Find number of terms when we expand $(a+b+c)^{99}$ (in the general case).

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5 Suppose that

$$S_n = \frac{5}{9} \times \frac{14}{20} \times \frac{27}{35} \times \cdots \times \frac{2n^2 - n - 1}{2n^2 + n - 1}$$

Find $\lim_{n \to \infty} S_n$.

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6 Let D and D' be two lines with the equations

$$\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{4} \quad \text{and} \quad \frac{x+1}{2} = \frac{y+2}{4} = \frac{z-1}{3}.$$

Find the length of their common perpendicular.

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7 Let B and C be two fixed point on the plane P. Find the locus of the points M on the plane P for which $MB^2 + kMC^2 = a^2$. (k and a are two given numbers and $k > 0$.)

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8 Define the operation $\bigoplus$ on the set of real numbers such that

$$x \bigoplus y = x + y - xy \qquad \forall x, y \in \mathbb{R}.$$

Prove that this operation is associative.