National Math Olympiad (Second Round) 1987
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Day 1

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1 Solve the following system of equations in positive integers

$$\begin{cases} a^3 - b^3 - c^3 = 3abc \\ \\ a^2 = 2(b+c) \end{cases}$$

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2 Let f be a real function defined in the interval $[0, +\infty)$ and suppose that there exist two functions f', f'' in the interval $[0, +\infty)$ such that

$$f''(x) = \frac{1}{x^2 + f'(x)^2 + 1} \qquad \text{and} \qquad f(0) = f'(0) = 0.$$

Let g be a function for which

$$g(0) = 0 \qquad \text{and} \qquad g(x) = \frac{f(x)}{x}.$$

Prove that g is bounded.

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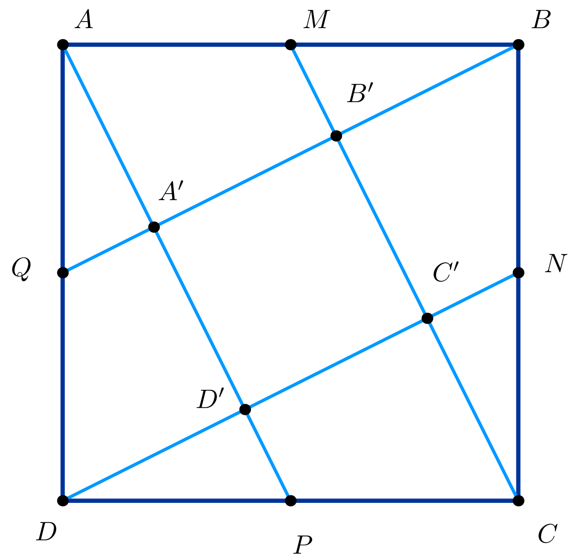
3 In the following diagram, let $ABCD$ be a square and let M, N, P and Q be the midpoints of its sides. Prove that

$$S_{A'B'C'D'} = \frac{1}{5} S_{ABCD}.$$



[S_X denotes area of the X.]

Day 2

1 Calculate the product:

$$A = \sin 1^\circ \times \sin 2^\circ \times \sin 3^\circ \times \cdots \times \sin 89^\circ$$

2 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = f(x)^2 + f(y)^2, \quad \forall x, y \in \mathbb{R}.$$

3 Let L_1, L_2, L_3, L_4 be four lines in the space such that no three of them are in the same plane. Let L_1, L_2 intersect in A, L_2, L_3 intersect in B and L_3, L_4 intersect in C. Find minimum and maximum number of lines in the space that intersect L_1, L_2, L_3 and L_4. Justify your answer.