National Math Olympiad (Second Round) 1988

www.artofproblemsolving.com/community/c3872

by Amir Hossein

Day 1

1 **(a)** Prove that for all positive integers m, n we have

$$\sum_{k=1}^{n} k(k+1)(k+2)\cdots(k+m-1) = \frac{n(n+1)(n+2)\cdots(n+m)}{m+1}$$

(b) Let $P(x)$ be a polynomial with rational coefficients and degree m. If n tends to infinity, then prove that

$$\frac{\sum_{k=1}^{n} P(k)}{n^{m+1}}$$

Has a limit.

2 In a cyclic quadrilateral $ABCD$, let I, J be the midpoints of diagonals AC, BD respectively and let O be the center of the circle inscribed in $ABCD$. Prove that I, J and O are collinear.

3 Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying

$$f(f(m) + f(n)) = m + n, \quad \forall m, n \in \mathbb{N}.$$

Prove that $f(x) = x$ for all $x \in \mathbb{N}$.

Day 2

1 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_1 = \frac{1}{2}$ and

$$a_n = \left(\frac{2n-3}{2n}\right) a_{n-1} \qquad \forall n \geq 2.$$

Prove that for every positive integer n, we have $\sum_{k=1}^{n} a_k < 1$.

2 In tetrahedron $ABCD$ let h_a, h_b, h_c and h_d be the lengths of the altitudes from each vertex to the opposite side of that vertex. Prove that

$$\frac{1}{h_a} < \frac{1}{h_b} + \frac{1}{h_c} + \frac{1}{h_d}.$$

3 Let n be a positive integer. 1369^n positive rational numbers are given with this property: if we remove one of the numbers, then we can divide remain numbers into 1368 sets with equal number of elements such that the product of the numbers of the sets be equal. Prove that all of the numbers are equal.